Classification of Fractals in Sierpinski Gasket Variation

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Abstract

In fractal science, Sierpinski gasket is well known, and Sierpinski gasket variation has been studied by H.-O. Peitgen et al. [6]. In this paper we algebraically prove the classification of fractals in Sierpinski gasket variation.

1. Sierpinski Gasket Variation

In xy plane we consider the square $S=\{(x, y) ; -w \leqq x \leqq w, -w \leqq y \leqq w\}$, where $w>0$. Let S_1, S_2, S_3, S_4 be the sub-squares of S, such that

$S_1=\{(x, y) ; -w \leqq x \leqq 0, -w \leqq y \leqq 0\}$, $S_2=\{(x, y) ; 0 \leqq x \leqq w, -w \leqq y \leqq 0\}$,

$S_3=\{(x, y) ; -w \leqq x \leqq 0, 0 \leqq y \leqq w\}$, $S_4=\{(x, y) ; 0 \leqq x \leqq w, 0 \leqq y \leqq w\}$.

Let d_0, d_1, d_2, d_3 be the rotations in S at the origin by $\theta=0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ radians, respectively. And let d_4, d_5, d_6, d_7 be the reflection mappings in S for the line $l_1 : y=0$, $l_2 : x=0$, $l_3 : y=x$, $l_4 : y=-x$, respectively.

The reduction mappings $v_i : S \rightarrow S_i$ for $i=1, 2, 3$ are defined by the following

$$v_1\begin{pmatrix} x \\ y \end{pmatrix}=\frac{1}{2}\begin{pmatrix} x \\ y \end{pmatrix}+\frac{w}{2}\begin{pmatrix} -1 \\ -1 \end{pmatrix}, \quad v_2\begin{pmatrix} x \\ y \end{pmatrix}=\frac{1}{2}\begin{pmatrix} x \\ y \end{pmatrix}+\frac{w}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad v_3\begin{pmatrix} x \\ y \end{pmatrix}=\frac{1}{2}\begin{pmatrix} x \\ y \end{pmatrix}+\frac{w}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

We shall obtain the matrix representation of $v_i d_k$ where $v_i d_k : S \rightarrow S_i$. Let

$$v_i d_k\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}+\begin{pmatrix} e \\ f \end{pmatrix}.$$

For $i=1$ we have the following table 1.

Table 1.

Map	a	b	c	d
$v_1 d_0$	1/2	0	0	1/2
$v_1 d_1$	0	−1/2	1/2	0
$v_1 d_2$	−1/2	0	0	−1/2
$v_1 d_3$	0	1/2	−1/2	0
$v_1 d_4$	1/2	0	0	−1/2
$v_1 d_5$	−1/2	0	0	1/2
$v_1 d_6$	0	1/2	1/2	0
$v_1 d_7$	0	−1/2	−1/2	0

And it always holds that $e=-w/2$, $f=-w/2$.

For $i=2, 3$ it follows that

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平成6年9月26日受理

$v_2d_k\binom{x}{y}=v_1d_k\binom{x}{y}+\frac{w}{2}\binom{1}{-1}$, $v_3d_k\binom{x}{y}=v_1d_k\binom{x}{y}+\frac{w}{2}\binom{-1}{1}$.

We note that d_k ($k=0, 1,2, \cdots, 7$) form a finite group. In fact we have the following composition table (Table 2), which shows the results of the composition d_kd_l.

Table 2.

l \ k	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	0	7	6	4	5
2	2	3	0	1	5	4	7	6
3	3	0	1	2	6	7	5	4
4	4	6	5	7	0	2	1	3
5	5	7	4	6	2	0	3	1
6	6	5	7	4	3	1	0	2
7	7	4	6	5	1	3	2	0

Moreover, we note below the relations that will be used later

$d_6d_0d_6=d_0$, $d_6d_1d_6=d_3$, $d_6d_2d_6=d_2$, $d_6d_3d_6=d_1$, $d_6d_4d_6=d_5$, $d_6d_5d_6=d_4$, $d_6d_6d_6=d_6$, $d_6d_7d_6=d_7$.

Now we can define the fractal set A. Let A_0 be the triangle in S with the three vertices $(-w, -w)$, $(w, -w)$, $(-w, w)$. The sequence $\{A_i\}$ is defined by

$A_i=v_1d_k(A_{i-1})\cup v_2d_l(A_{i-1})\cup v_3d_m(A_{i-1})$ $(i=1, 2, 3, \cdots)$,

where k, l, m is any number in $\{0, 1, 2, \cdots, 7\}$, respectively. And define $A=\lim_{i\to\infty} A_i$. Then we find that A is the fractal with the fractal dimension $\log 3/\log 2$.

We call that the fractal A is generated by the patten $<k, l, m>$, or we denote $A=<k, l, m>$. Let $\mathscr{A}$ be the set $\{<k, l, m>; k, l, m=0, 1, 2, \cdots, 7\}$. The number of elements in $\mathscr{A}$ is 512 as patterns. We note that the fractal A with the pattern $<k, l, m>$ satisfies $A=v_1d_k(A)\cup v_2d_l(A)\cup v_3d_m(A)$. On the contrary, if $A\in\mathscr{A}$ satisfies $A=v_1d_k(A)\cup v_2d_l(A)\cup v_3d_m(A)$, then A is the fractal which is generated by the pattern $<k, l, m>$.

2. Classification of Patterns

In $\mathscr{A}$, there are some identical patterns or some pairs of patterns which one of the pair is the symmetric transformation of the other. For example, pattern $<0, 0, 0>$ and pattern $<0, 6, 0>$ are the identical fractals, or $<0, 0, 1>$ is the symmetric transformation with respect to the line $y=x$ of $<0, 3, 0>$. In this paper we want to make the classfication of $\mathscr{A}$.

Lemma 1.

For $k\neq 0, 6$ there do not exist $A, B\in\mathscr{A}$ such that $d_k(A)=B$, especially there does not exist $A\in\mathscr{A}$ such that $d_k(A)=A$.

Proof.

Let $A=V_1\cup V_2\cup V_3\cup V_4$, $V_i\subset S_i$, $B=W_1\cup W_2\cup W_3\cup W_4$, $W_i\subset S_i$.

Then $V_1\neq\emptyset$, $V_2\neq\emptyset$, $V_3\neq\emptyset$, $V_4=\emptyset$, $W_1\neq\emptyset$, $W_2\neq\emptyset$, $W_3\neq\emptyset$, $W_4=\emptyset$.

If $d_k(A)=B$ for $\theta=\frac{\pi}{2}, \pi, \frac{3\pi}{2}$ then $W_4\neq\emptyset$. It is a contradiction. Also, in case of symmetry with the line $l_1: y=0$, $l_2: x=0$, $l_4: y=-x$, we have $W_4\neq\emptyset$, this is a contradiction. … Q.E.D.

If there exist $A, B\in\mathscr{A}$ such that $d_k(A)=B$, then A is some transformation of B, so we may think that A and B are almost the same fractals. *Lemma* 1 implies that the transformation is the symmetric transformation with respect to the line $y=x$ only. So we have the next definition.

Definition 1.

Let $A=<k, l, m>$, $B=<s, t, u>$. We denote $<k, l, m>\sim<s, t, u>$ if and only if $d_6(A)=B$.

The next lemma is contrapositive of the second part in *Lemma* 1.

Lemma 2.

If there exists $A\in\mathscr{A}$ such that $d_k(A)=A$, then $k=0,6$.

If $A\in\mathscr{A}$ has some symmetricities itself, then the symmetry is only with respect to the line $y=x$. So we use the word 'symmetric' in this mean only.

Lemma 3.

We consider $A=<k, l, m>$. If there exists $(s, t, u)\neq(k, l, m)$ such that $<s, t, u>=<k, l, m>$, then A is symmetric.

Proof.

We have $A=v_1d_k(A)\cup v_2d_l(A)\cup v_3d_m(A)=v_1d_s(A)\cup v_2d_t(A)\cup v_3d_u(A)$.

Since $v_i: S\rightarrow S_i$ is one to one mappings,

$d_k(A)=d_s(A)$, $d_l(A)=d_t(A)$, $d_m(A)=d_u(A)$.

So $d_s^{-1}d_k(A)=A$. If we assume that A is not symmetric, then from *Lemma* 2 we have $d_s^{-1}d_k=d_0$ only. Hence $s=k$. In the same way we have $t=l$, $u=m$. This is a contradiction of the asummption. … Q.E.D.

Considering the contrapositive of *Lemma* 3, if $A\in\mathscr{A}$ is not symmetric then A has no other patterns which equal to A, that is, the multiplicity of A is only one.

Lemma 4.

$d_6v_2=v_3d_6$, $d_6v_3=v_2d_6$, $d_6v_1=v_1d_6$.

Proof.

We note that $d_6v_2: S\rightarrow S_3$, $d_6v_3: S\rightarrow S_2$, $v_3d_6: S\rightarrow S_3$, $v_2d_6: S\rightarrow S_2$.

The matrix representation of the mappings is the following

$$v_3d_6\begin{pmatrix}x\\y\end{pmatrix}=\frac{1}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\frac{w}{2}\begin{pmatrix}-1\\1\end{pmatrix},$$

$$d_6v_2\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\frac{x}{2}+\frac{w}{2}\\\frac{y}{2}-\frac{w}{2}\end{pmatrix}=\frac{1}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\frac{w}{2}\begin{pmatrix}-1\\1\end{pmatrix}.$$

Hence we have $d_6v_2=v_3d_6$. And other cases are the same. … Q.E.D.

Lemma 5.

We consider $A=<k, l, m>$, and define $d_s\equiv d_6d_kd_6$, $d_t\equiv d_6d_ld_6$, $d_u\equiv d_6d_md_6$, then we have $<k, l, m>\sim<s, u, t>$.

Proof.

$A=v_1d_k(A)\cup v_2d_l(A)\cup v_3d_m(A)$. From *Lemma* 4, we have

$$\begin{aligned}d_6(A)&=d_6v_1d_k(A)\cup d_6v_2d_l(A)\cup d_6v_3d_m(A)\\&=v_1d_6d_k(A)\cup v_3d_6d_l(A)\cup v_2d_6d_m(A)\\&=v_1d_6d_kd_6(d_6(A))\cup v_2d_6d_md_6(d_6(A))\cup v_3d_6d_ld_6(d_6(A))\\&=v_1d_s(d_6(A))\cup v_2d_u(d_6(A))\cup v_3d_t(d_6(A)).\end{aligned}$$

It shows that $d_6(A)$ is generated by the pattern $<s, u, t>$. … Q.E.D.

Remark 1.

In *Lemma* 5, we remark that $<s, u, t>$, is not $<s, t, u>$.

Also, *Lemma* 5 shows that the symmetric transformed pattern of $A=<k, l, m>$ is given when k, l, m transform to s, t, u by the next rules,

$0 \to 0$; $1 \to 3$; $2 \to 2$; $3 \to 1$; $4 \to 5$; $5 \to 4$; $6 \to 6$; $7 \to 7$

and substitute t for u, and u for t, thus we have the symmetric transformed pattern $<s, u, t>$.

Now we consider how patterns are symmetric.

Lemma 6.

Let $A = <k, l, m>$ be symmetric, then it must be $k = 0, 2, 6, 7$ and

$(l, m) \in \{(0,0), (0,6), (1,3), (1,4), (2,2), (2,7), (3,1), (3,5), (4,1), (4,5), (5,3), (5,4), (6,0), (6,6)\ (7,2), (7,7)\}$.

Proof.

$A = \nu_1 d_k(A) \cup \nu_2 d_l(A) \cup \nu_3 d_m(A)$ is symmetric, so $\nu_1 d_k(A)$ is also symmetric. Hence $d_k(A)$ is symmetric, that is, $d_6 d_k(A) = d_k(A)$. From *Lemma* 2 we obtain that $d_6 d_k = d_k$ or $d_6 d_k = d_k d_6$. Since $d_6 d_k = d_k$ is impossible, we have $d_6 d_k = d_k d_6$ only. So we get $k = 0, 2, 6, 7$.

Next, we shall prove the second part of the lemma.

Let $d_s = d_6 d_k d_6$, $d_t = d_6 d_l d_6$, $d_u = d_6 d_m d_6$. From *Lemma* 5, it follows that $<s, u, t> = d_6(A)$. Since $A = <k, l, m>$ is symmetric, we have $<k, l, m> = <s, u, t>$. Hence, $\nu_1 d_k(A) \cup \nu_2 d_l(A) \cup \nu_3 d_m(A) = \nu_1 d_s(A) \cup \nu_2 d_u(A) \cup \nu_3 d_t(A)$. So we have $d_k(A) = d_s(A)$, $d_l(A) = d_u(A)$, $d_m(A) = d_t(A)$. From *Lemma* 2, it follows that $d_l = d_u$ or $d_l = d_u d_6$. Also we already have $d_m = d_6 d_u d_6$. For $u = 0, 1, 2, \cdots, 7$ we can calculate (l, m) in the lemma. … Q.E.D.

From *Lemma* 6 we find that the number of symmetric patterns in $\mathscr{A}$ is $4 \times 6 = 64$. We want to know how patterns are identical in these symmetric patterns.

Lemma 7.

Let $d_k(A) = d_s(A)$ for $A \in \mathscr{A}$, then it must be

$(k, s) \in \boldsymbol{E} \equiv \{(0,0), (0,6), (1,1), (1,5), (2,2), (2,7), (3,3), (3,4), (4,4), (4,3), (5,5), (5,1), (6,6), (6,0), (7,7), (7,2)\}$.

Proof.

From *Lemma* 2 it follows that $d_k^{-1} d_s = d_0$ or $d_k^{-1} d_s = d_6$. Hence we have $d_s = d_k$ or $d_s = d_k d_6$. For $k = 0, 1, 2, \cdots, 7$ we can calculate (k, s) in the lemma. … Q.E.D.

Lemma 8.

Let $A = <k, l, m>$ be symmetric. Then $<s, t, u> = <k, l, m>$ is equivalent to $(k, s), (l, t), (m, u) \in \boldsymbol{E}$, where $\boldsymbol{E}$ is the set which is defined in *Lemma* 7.

Proof.

Lemma 7 implies that $(k, s), (l, t), (m, u) \in \boldsymbol{E}$ is equivalent to $d_s(A) = d_k(A)$, $d_t(A) = d_l(A)$, $d_u(A) = d_m(A)$.

Hence $A = \nu_1 d_k(A) \cup \nu_2 d_l(A) \cup \nu_3 d_m(A) = \nu_1 d_s(A) \cup \nu_2 d_t(A) \cup \nu_3 d_u(A)$.

It shows that A is generated by the pattern $<s, t, u>$. It follows that $<s, t, u> = <k, l, m>$. Also the inverse property is true. … Q.E.D.

Remark 2.

$(k, s), (l, t), (m, u) \in \boldsymbol{E}$ means that k, l, m transform to s, t, u in the next rules

$0 \to 0, 6$; $1 \to 1, 5$; $2 \to 2, 7$; $3 \to 3, 4$; $4 \to 3, 4$; $5 \to 1, 5$; $6 \to 0, 6$; $7 \to 2, 7$.

For example if $k = 0$ then $s = 0$ or 6, if $l = 4$ then $t = 3$ or 4 etc.

Also we find that if $A = <k, l, m>$ is symmetric, then the multiplicity of A is 8, because k, l, m are transformed in two ways, respectively.

Theorem 1.

The identical patterns in $\mathscr{A}$ are the following only

$<0,0,0> = <0,0,6> = <0,6,0> = <0,6,6> = <6,0,0> = <6,0,6> = <6,6,0> = <6,6,6>$,

$<0,1,3> = <0,1,4> = <0,5,3> = <0,5,4> = <6,1,3> = <6,1,4> = <6,5,3> = <6,5,4>$,

$<0,2,2> = <0,2,7> = <0,7,2> = <0,7,7> = <6,2,2> = <6,2,7> = <6,7,2> = <6,7,7>$,

<0,3,1>=<0,3,5>=<0,4,1>=<0,4,5>=<6,3,1>=<6,3,5>=<6,4,1>=<6,4,5>,
<2,0,0>=<2,0,6>=<2,6,0>=<2,6,6>=<7,0,0>=<7,0,6>=<7,6,0>=<7,6,6>,
<2,1,3>=<2,1,4>=<2,5,3>=<2,5,4>=<7,1,3>=<7,1,4>=<7,5,3>=<7,5,4>,
<2,2,2>=<2,2,7>=<2,7,2>=<2,7,7>=<7,2,2>=<7,2,7>=<7,7,2>=<7,7,7>,
<2,3,1>=<2,3,5>=<2,4,1>=<2,4,5>=<7,3,1>=<7,3,5>=<7,4,1>=<7,4,5>.

Proof.

Let $A=<k, l, m>$. If there exist other patterns which are the same as A, then from *Lemma* 3 it follows that A is symmetric. *Lemma* 6 implies that the number of symmetric patterns in $\mathscr{A}$ are 64. In these symmetric patterns we can find the same patterns by the rules of *Lemma* 8. For example, we find that the 8 patterns are the same as <0,0,0>, and in the remainder we find the 8 same patterns in a like manner. So we conclude this theorem. ··· Q.E.D.

Finally, we state the program of Sierpinski gasket variation, the program is already introduced by H.-O. Peitgen et al. [6]. But we improved the program so that it can be easily applied to any patterns. We wrote the program in BASIC, using Fujitsu F-BASIC86HG.

BASIC Program

```
10  rem *** Sierpinski Gasket Variation ***
20   dim A(3), B(3), C(3), D(3), E(3), F(3)
30   dim XLEFT(10), YLEFT(10), XRIGHT(10), YRIGHT(10), XTOP(10), YTOP(10)
40   cls : view (450, 150) - (900, 600)
50   W=100 : window (-W,-W) - (W, W)
60  rem Input of Pattern and Level
70   locate 32, 2 : print "Sierpinski Gasket Variation"
80   locate 1, 8 : print "Pattern=" : locate 1, 10 : print "Level (1～8) ="
90   locate 1, 15 : print "For example," : locate 1, 16 : print "Pattern=0, 2, 7"
100  locate 10, 8 : input P1, P2, P3
110  locate 14, 10 : input LEVEL
120 rem Set the Initial Values
130  XLEFT(LEVEL) =-W : YLEFT(LEVEL) =-W : XRIGHT(LEVEL) =W
140  YRIGHT(LEVEL) =-W : XTOP(LEVEL) =-W : YTOP(LEVEL) =W
150  gosub *DATA
160 rem Main Caluculation
170  gosub 260
180  end
190 rem Transformation of Triangle
200  XLEFT(LEVEL) =A(M) * XLEFT(LEVEL+1) +B(M) * YLEFT(LEVEL+1) +E(M)
210  YLEFT(LEVEL) =C(M) * XLEFT(LEVEL+1) +D(M) * YLEFT(LEVEL+1) +F(M)
220  XRIGHT(LEVEL) =A(M) * XRIGHT(LEVEL+1) +B(M) * YRIGHT(LEVEL+1) +E(M)
230  YRIGHT(LEVEL) =C(M) * XRIGHT(LEVEL+1) +D(M) * YRIGHT(LEVEL+1) +F(M)
240  XTOP(LEVEL) =A(M) * XTOP(LEVEL+1) +B(M) * YTOP(LEVEL+1) +E(M)
250  YTOP(LEVEL) =C(M) * XTOP(LEVEL+1) +D(M) * YTOP(LEVEL+1) +F(M)
260 rem Draw the Least Triangle
270  if LEVEL>1 then goto 320
```

```
280  line (XLEFT(1),-YLEFT(1))-(XRIGHT(1),-YRIGHT(1)), pset,15,,
290  line -(XTOP(1),-YTOP(1)), pset,15,,
300  line -(XLEFT(1),-YLEFT(1)), pset,15,,
310  goto 410
320 rem Branch into the Lower Triangle
330  LEVEL=LEVEL-1
340  M=1
350  gosub 190
360  M=2
370  gosub 190
380  M=3
390  gosub 190
400  LEVEL=LEVEL+1
410  return
420 * DATA
430  for I=0 to 7
440   read A, B, C, D
450   if I><P1 then goto 470
460   A(1)=A : B(1)=B : C(1)=C : D(1)=D
470   if I><P2 then goto 490
480   A(2)=A : B(2)=B : C(2)=C : D(2)=D
490   if I><P3 then goto 510
500   A(3)=A : B(3)=B : C(3)=C : D(3)=D
510  next I
520  E(1)=-W/2 : F(1)=-W/2 : E(2)=W/2 : F(2)=-W/2 : E(3)=-W/2 : F(3)=W/2
530  data 0.5, 0, 0, 0.5
540  date 0,-0.5, 0.5, 0
550  data -0.5, 0, 0,-0.5
560  data 0, 0.5,-0.5, 0
570  data 0.5, 0, 0,-0.5
580  data -0.5, 0, 0, 0.5
590  data 0, 0.5, 0.5, 0
600  data 0,-0.5,-0.5, 0
610 return
```

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