〔論　文〕

正多面体と半正多面体の変形図式

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Deformation Scheme of Regular and Semiregular Polyhedra

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Abstract

We view 5 types of regular polyhedra (Platonic solids) and 13 types of semiregular polyhedra (Archimedean solids) as graphs with vertices and edges in the meaning of the graph theory; consider deformations of these graphs under the condition stemming from chemistry, that is, no vertices and edges are lost except for the confluence of vertices, and are yielded except for by splitting the vertices; and establish that all regular polyhedra and all semiregular polyhedra are connected by deformations by concretely showing these deformations. Along such a deformation, it has already been shown that the best constant of Sobolev inequality on a truncated tetrahedron is reduced to one on a regular tetrahedron with a simple energy function. It is conjectured that deformation extends not only to the graphs but also to the discrete harmonic analytic structures of all regular and semiregular polyhedra, one of which is homotopic to C60 buckyball.

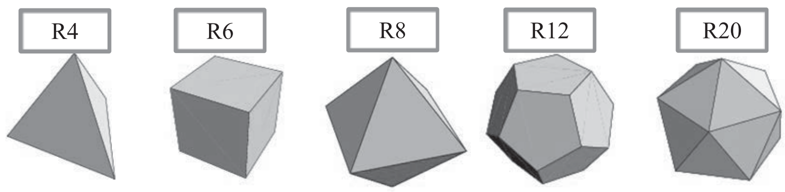
Keywords：Regular polyhedron, Semiregular polyhedron, Graph, C60 buckyball

1．Introduction

1・1　Regular and semiregular polyhedra

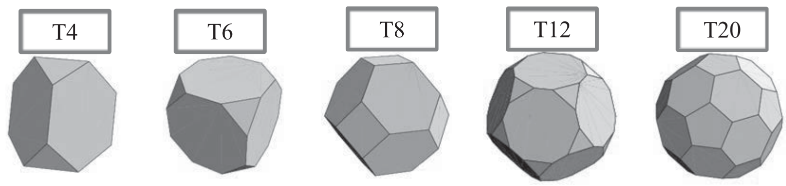
Regular polyhedra are extremely mathematical objects. Each one is a solid with faces of a single type of regular polygon. The regular polyhedron with n faces is denoted as Rn. Only 5 types of regular polyhedra exist as follows [1]:

R 4: Regular tetrahedron
R 6: Cube (=Regular hexahedron)
R 8: Regular octahedron
R 12: Regular dodecahedron
R 20: Regular icosahedron



Semiregular polyhedra are also called Archimedean solids. Each one is a solid with faces of several types of regular polygon. There are 13 semiregular solids [1], which are classified into four types as follows:

(i) Truncated polyhedra:
T 4: Truncated tetrahedron
T 6: Truncated cube
T 8: Truncated octahedron
T 12: Truncated dodecahedron
T 20: Truncated icosahedron



(ii) Snub polyhedra:
S 6: Snub cube
S 12: Snub dodecahedron
(iii) Quasi-regular polyhedra:
Q 6-8: Cuboctahedron
Q 20-12: Icosidodecahedron

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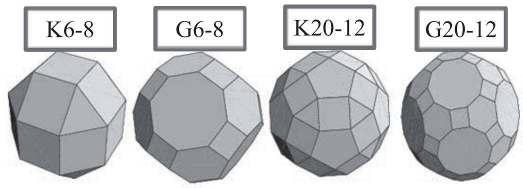
平成28年12月７日受理

(iv) Rhombic *n*-*m*-hedra:
K 6-8: Small rhombicuoctahedron (=Rhombicuboctahedron)
G 6-8: Great rhombicuboctahedron (=Rhombitruncated cuboctahedron)
K 20-12: Small rhombicosidodecahedron (=Rhombicosidodecahedron)
G 20-12: Great rhombicosidodecahedron
(=Rhombitruncated icosidodecahedron)



In the above, all the pictures of regular or semiregular polyhedra are from *WolframMathWorld* [2].

For each regular or semiregular polyhedron, let us denote the number of vertices, edges, and faces by V, E, and F, respectively. We list these numbers: the reader can check whether the well-known relation $V-E+F=2$, which is called Euler's polyhedron formula, is valid [1].

Rn	V	E	F
R4	4	6	4
R6	8	12	6
R8	6	12	8
R12	20	30	12
R20	12	30	20

Tn	V	E	F
T4	12	18	8
T6	24	36	14
T8	24	36	14
T12	60	90	32
T20	60	90	32

Sn	V	E	F
S6	24	60	38
S12	60	150	92

Q/K/Gn-m	V	E	F
Q6-8	12	24	14
K6-8	24	48	26
G6-8	48	72	26
Q20-12	30	60	32
K20-12	60	120	62
G20-12	120	180	62

Euler's polyhedron formula
$$V-E+F=2$$

1・2　Background and known results

We introduce the works of Kametaka school and that of the corresponding author.

Kametaka et al. studied the best constant of the Sobolev inequality in view of the boundary value problem [3,4,5,6,7], and they studied discrete Sobolev inequality [8,9,10,11,12] to apply it to C60 buckyball [13]. The Sobolev inequality, known as the Sobolev embedding theorem, plays an important role in the theory of partial differential equations. Brezis [14, Chap.IX] gave a constant for the Sobolev inequality and remarked that the best constant was known and complex. Talenti [15] and Marti [16] studied the best constant using variational methods. The work of the Kametaka school on each polyhedron is performed under the assumption of uniformity of the spring constants.

In contrast, the chemistry of fullerenes studies its structure in detail [17]. According to [18,19,20], the bond lengths of C 60 buckyball are of two types. With regard to the application to the chemistry of fullerenes, the assumption of uniformity of the spring constants should not be considered.

In an article [21] concerning the best constant of discrete Sobolev inequality on T4 with two spring constants, in other words, a weighted T4 graph, the corresponding author generalized the results of the Kametaka school for R4 [9] and T4 [11] using continuous deformation with a parameter, i.e., the ratio of two spring constants. Deformation implies destruction of symmetry. High symmetry moves us by its beauty; however, the destruction of symmetry also fascinates us.

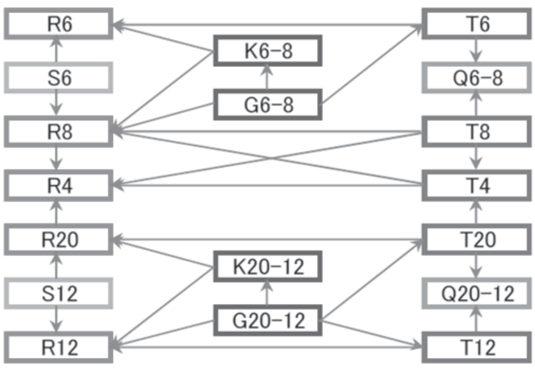
1・3　Results

We assume that the following continuity condition is valid: no vertices and edges are lost except for the confluence of vertices, and no vertices and edges are yielded except for by splitting the vertices. We consider the graph of vertices and edges of a polyhedron as a molecule. The continuity condition implies that the atoms do not abruptly appear or vanish.

In the article [21], the corresponding author deformed T4 into R4 continuously and studied the energy in a polyhedron. For example, R4 becomes T4 by the truncation of the corners, and more truncation finally yields R8. R4 can be deformed continuously under the continuity condition into T4, and then into R8. This fact suggests that some regular polyhedra are connected by continuous deformations satisfying the continuity condition, and some semiregular polyhedra are intermediate products of such deformations. Our main result is as follows.

Theorem. All regular polyhedra and semiregular polyhedra are connected by deformations satisfying the continuity condition.

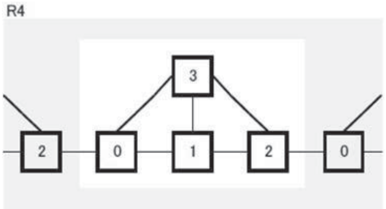
Instead of a proof, we give the scheme of coalescence of vertices. For example, T4 has 12 vertices, and along the deformation T4→R4, three vertices converge to one vertex of R4. To identify each vertex, we first give the numbering of the vertices of all regular and semiregular polyhedra, and then give the correspondence of the vertices along each deformation in the diagram to the right of the theorem. The direction of the arrow means coalescence, i.e., the decay of the number of vertices.



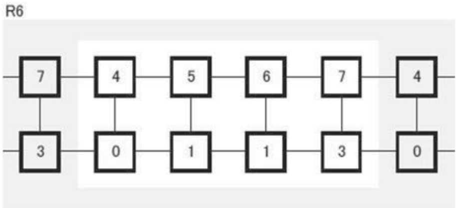
2. Numbering the vertices

First, we number of the vertices of the polyhedra R*n*, T*n*, S*n*, Q*m*-*n*, K*m*-*n*, and G*m*-*n* as follows:

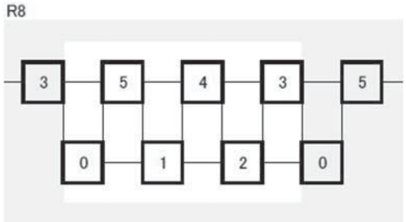
R4



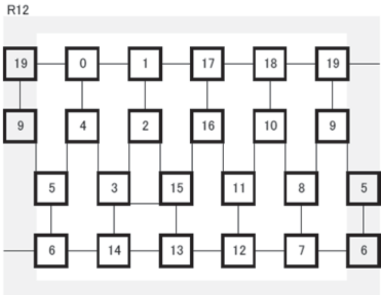
R6



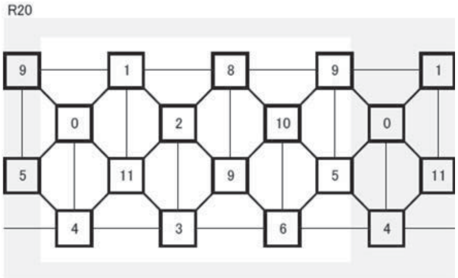
R8



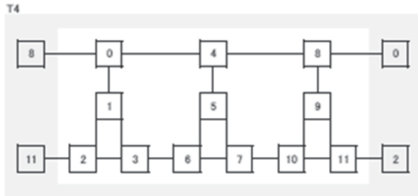
R12



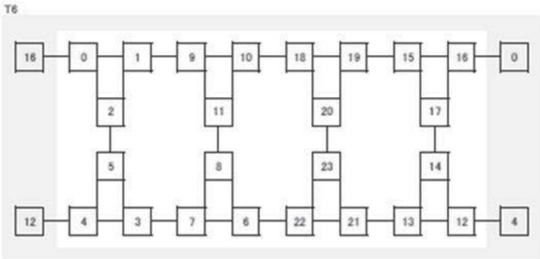
R20



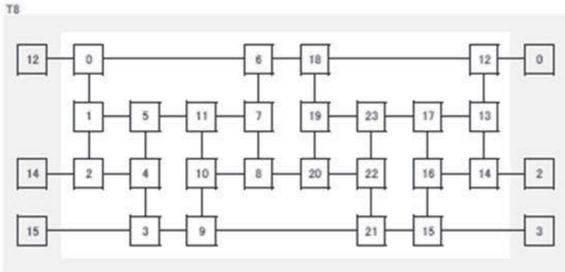
T4



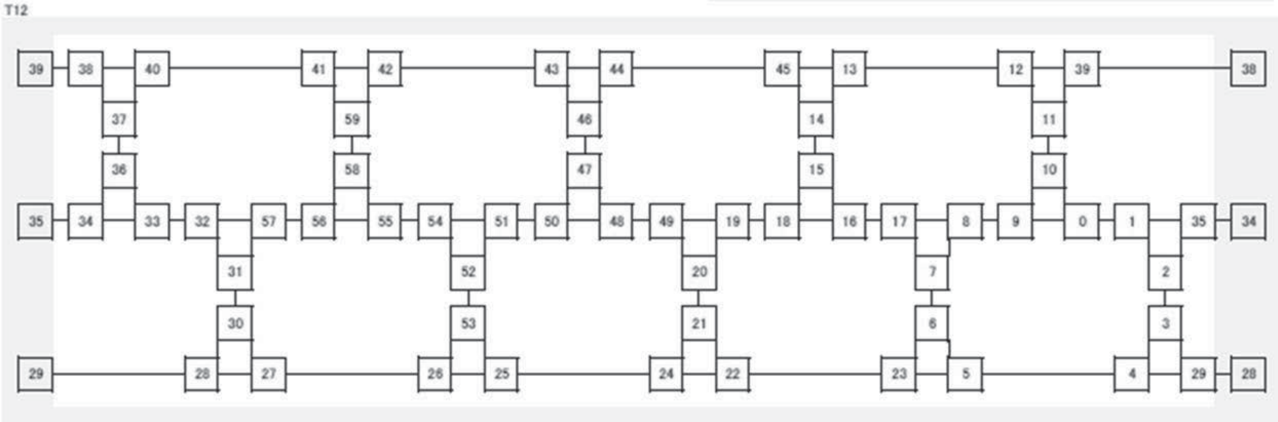
T6



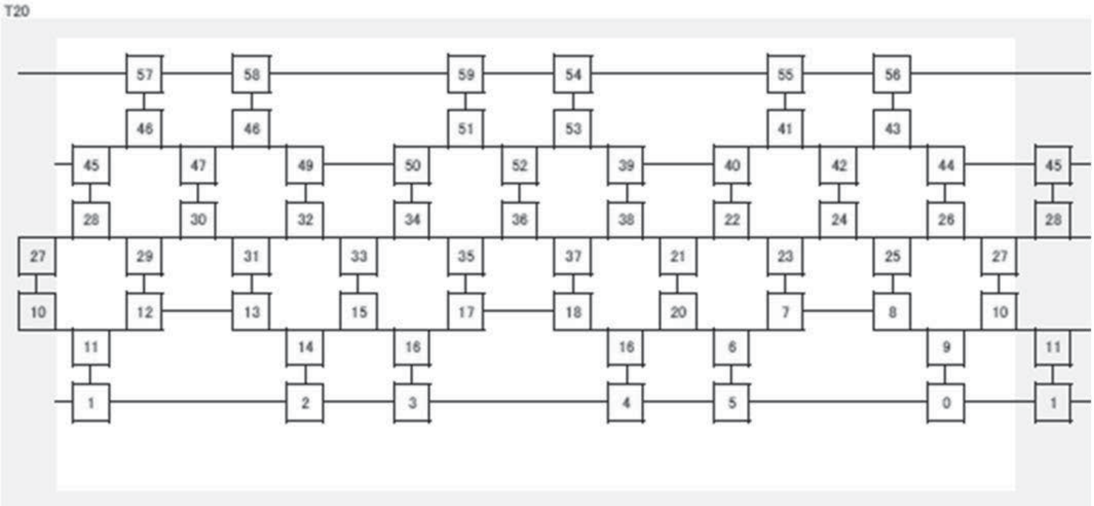
T8



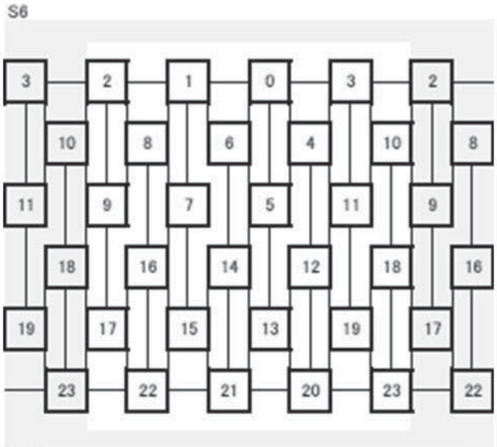
T12



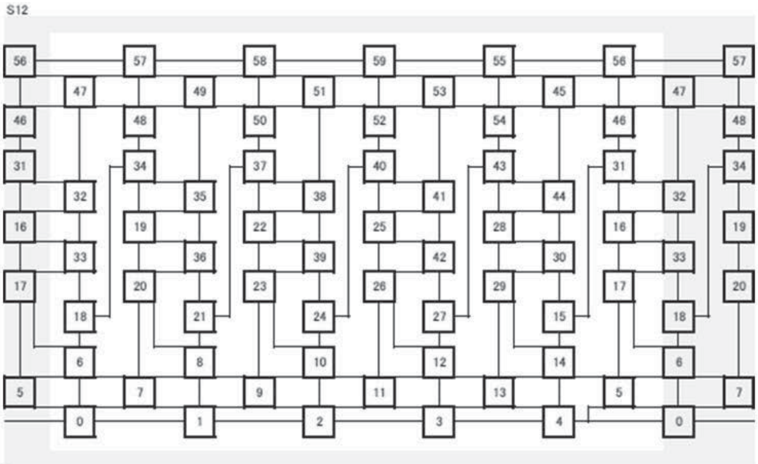
T20



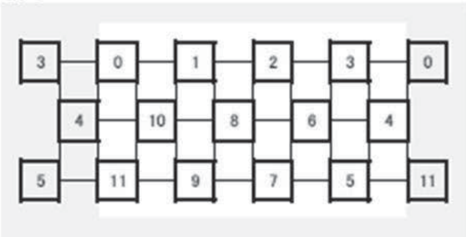
S6



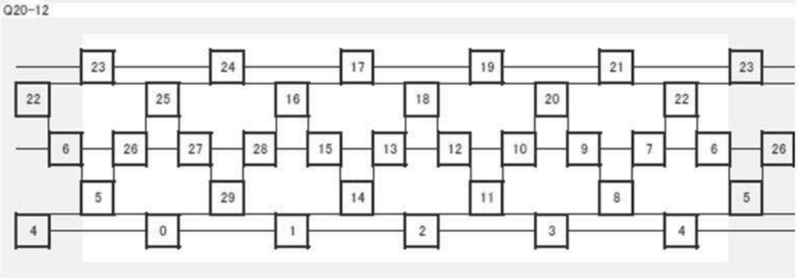
S12



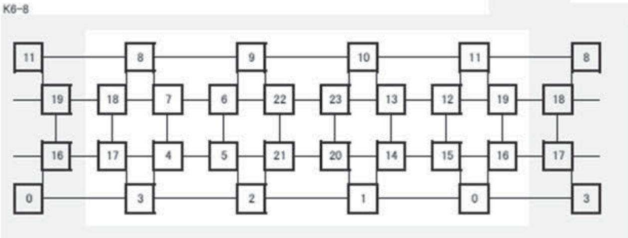
Q6-8



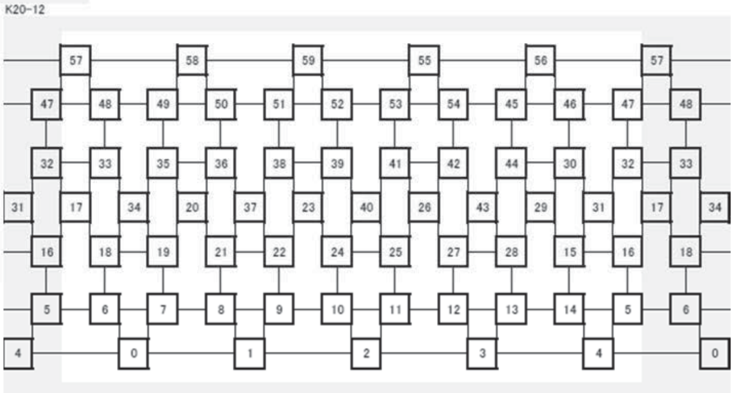
Q20-12



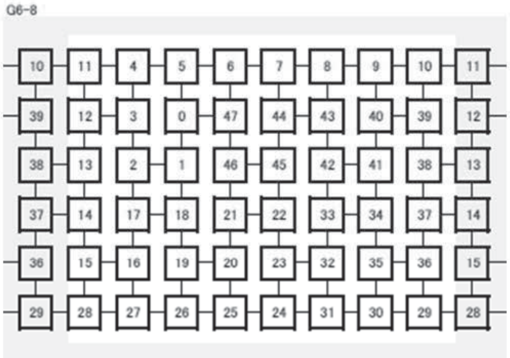
K6-8



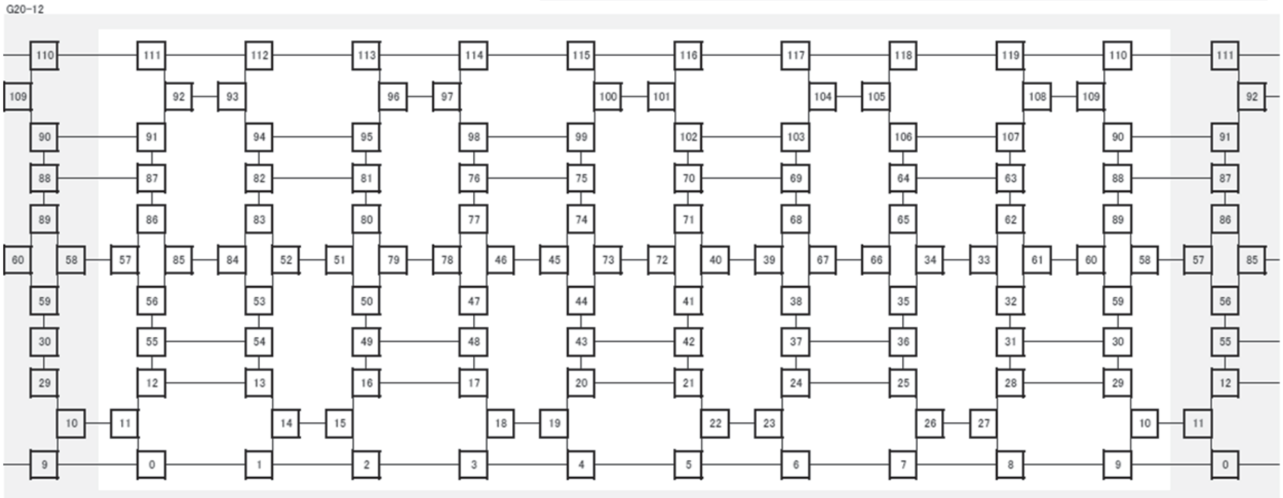
K20-12



G6-8



G20-12



3. Deformation scheme

Second, we give the correspondences for each deformation.

T4→R4

Vertices of T4		Vertices of R4
0, 1, 9	→	0
2, 3, 11	→	1
4, 5, 6	→	2
7, 8, 10	→	3

T12→R12

Vertices of T12		Vertices of R12
0, 9, 10	→	0
1, 2, 35	→	1
3, 4, 29	→	2
5, 6, 23	→	3
7, 8, 17	→	4
15, 16, 18	→	5
19, 20, 49	→	6
47, 48, 50	→	7
43, 44, 46	→	8
13, 14, 45	→	9
41, 42, 59	→	10
55, 56, 58	→	11
51, 52, 54	→	12
25, 26, 53	→	13
21, 22, 24	→	14
27, 28, 30	→	15
31, 32, 57	→	16
33, 34, 36	→	17
37, 38, 40	→	18
11, 12, 39	→	19

T8→R4

Vertices of T8		Vertices of R4
0, 1, 2, 3, 4, 5	→	0
16, 17, 18, 19, 20, 23	→	1
10, 11, 12, 13, 14, 15	→	2
6, 7, 8, 9, 21, 22	→	3

T4→R8

Vertices of T4		Vertices of R8
0, 5	→	0
1, 2	→	1
3, 4	→	2
6, 7	→	3
10, 11	→	4
8, 9	→	5

T8→T4

Vertices of T8		Vertices of T4
1, 18	→	0
2, 13	→	1
3, 12	→	2
4, 7	→	3
5, 6	→	4
0, 19	→	5
20, 21	→	6
22, 23	→	7
15, 16	→	8
14, 17	→	9
9, 10	→	10
8, 11	→	11

T6→Q6-8

Vertices of T6		Vertices of Q6-8
0, 1	→	0
2, 3	→	1
4, 5	→	2
6, 7	→	3
8, 9	→	4
10, 11	→	5
12, 13	→	6
14, 15	→	7
16, 17	→	8
18, 19	→	9
20, 21	→	10
22, 23	→	11

T6→R6

Vertices of T6		Vertices of R6
0, 7, 8	→	0
1, 2, 21	→	1
3, 4, 17	→	2
5, 6, 13	→	3
9, 10, 23	→	4
19, 20, 22	→	5
15, 16, 18	→	6
11, 12, 14	→	7

T20→R20

Vertices of T20		Vertices of R20
0, 1, 9, 10, 11	→	0
2, 3, 14, 15, 16	→	1
17, 18, 35, 36, 37	→	2
21, 22, 38, 39, 40	→	3
7, 8, 23, 24, 25	→	4
26, 27, 28, 44, 45	→	5
41, 42, 43, 55, 56	→	6
51, 52, 53, 54, 59	→	7
32, 33, 34, 49, 50	→	8
12, 13, 29, 30, 31	→	9
46, 47, 48, 57, 58	→	10
4, 5, 6, 19, 20	→	11

S6→R6

Vertices of S6		Vertices of R6
0, 5, 6	→	0
1, 7, 8	→	1
2, 9, 10	→	2
3, 4, 11	→	3
12, 19, 20	→	4
13, 14, 21	→	5
15, 16, 22	→	6
17, 18, 23	→	7

S6→R8

Vertices of S6		Vertices of R8
0, 1, 2, 3	→	0
4, 5, 12, 13	→	1
10, 11, 18, 19	→	2
8, 9, 16, 17	→	3
20, 21, 22, 23	→	4
6, 7, 14, 15	→	5

S12→R20

Vertices of S12		Vertices of R20
0, 1, 2, 3, 4	→	0
5, 14, 15, 16, 17	→	1
32, 33, 34, 47, 48	→	2
35, 36, 37, 49, 50	→	3
8, 9, 21, 22, 23	→	4
10, 11, 24, 25, 26	→	5
38, 39, 40, 51, 52	→	6
55, 56, 57, 58, 59	→	7
30, 31, 44, 45, 46	→	8
12, 13, 27, 28, 29	→	9
41, 42, 43, 53, 54	→	10
6, 7, 18, 19, 20	→	11

T8→Q6-8

Vertices of T8		Vertices of Q6-8
0, 5	→	0
1, 2	→	1
17, 18	→	2
19, 20	→	3
6, 21	→	4
9, 22	→	5
16, 23	→	6
10, 15	→	7
13, 14	→	8
11, 12	→	9
3, 4	→	10
7, 8	→	11

T8→R8

Vertices of T8		Vertices of R8
0, 1, 18, 19	→	0
4, 5, 6, 7	→	1
20, 21, 22, 23	→	2
14, 15, 16, 17	→	3
8, 9, 10, 11	→	4
2, 3, 12, 13	→	5

R8→R4

Vertices of R8		Vertices of R4
0	→	0
1	→	1
2	→	2
3, 4, 5	→	3

R20→R4

Vertices of R20		Vertices of R4
0, 5, 9	→	0
6, 7, 10	→	1
3, 4, 11	→	2
1, 2, 8	→	3

T20→T4

Vertices of T20		Vertices of T4
0, 1, 9, 10, 11	→	0
2, 3, 14, 15, 16	→	1
32, 33, 34, 49, 50	→	2
12, 13, 29, 30, 31	→	3
26, 27, 28, 44, 45	→	4
7, 8, 23, 24, 25	→	5
41, 42, 43, 55, 56	→	6
21, 22, 38, 39, 40	→	7
17, 18, 35, 36, 37	→	8
4, 5, 6, 19, 20	→	9
51, 52, 53, 54, 59	→	10
46, 47, 48, 57, 58	→	11

S12→R12

Vertices of S12		Vertices of R12
0, 5, 6	→	0
1, 7, 8	→	1
2, 9, 10	→	2
3, 11, 12	→	3
4, 13, 14	→	4
15, 29, 30	→	5
28, 43, 44	→	6
45, 54, 55	→	7
46, 47, 56	→	8
16, 31, 32	→	9
48, 49, 57	→	10
50, 51, 58	→	11
52, 53, 59	→	12
25, 40, 41	→	13
26, 27, 42	→	14
23, 24, 39	→	15
22, 37, 38	→	16
20, 21, 36	→	17
19, 34, 35	→	18
17, 18, 33	→	19

K20-12→R12

Vertices of K20-12		Vertices of R12
0, 6, 7	→	0
1, 8, 9	→	1
2, 10, 11	→	2
3, 12, 13	→	3
4, 5, 14	→	4
15, 16, 31	→	5
29, 30, 44	→	6
45, 46, 56	→	7
47, 48, 57	→	8
17, 32, 33	→	9
49, 50, 58	→	10
51, 52, 59	→	11
53, 54, 55	→	12
26, 41, 42	→	13
27, 28, 43	→	14
24, 25, 40	→	15
23, 38, 39	→	16
21, 22, 37	→	17
20, 35, 36	→	18
18, 19, 34	→	19

Remark. Deformation G6-8→R8 can be obtained by the composition of two deformations: G6-8→K6-8 and K6-8→R8.

T20→Q20-12

Vertices of T20		Vertices of Q20-12
0, 5	→	0
8, 9	→	1
10, 27	→	2
11, 12	→	3
1, 2	→	4
3, 4	→	5
16, 17	→	6
15, 33	→	7
13, 14	→	8
31, 32	→	9
20, 47	→	10
28, 29	→	11
45, 46	→	12
43, 44	→	13
25, 26	→	14
24, 42	→	15
40, 41	→	16
54, 55	→	17
56, 57	→	18
58, 59	→	19
48, 49	→	20
50, 51	→	21
34, 35	→	22
36, 52	→	23
39, 53	→	24
37, 38	→	25
18, 19	→	26
20, 21	→	27
22, 23	→	28
6, 7	→	29

T12→Q20-12

Vertices of T12		Vertices of Q20-12
0, 1	→	0
2, 3	→	1
4, 5	→	2
6, 7	→	3
8, 9	→	4
10, 11	→	5
12, 13	→	6
14, 15	→	7
16, 17	→	8
18, 19	→	9
20, 21	→	10
22, 23	→	11
24, 25	→	12
26, 27	→	13
28, 29	→	14
30, 31	→	15
56, 57	→	16
54, 55	→	17
52, 53	→	18
50, 51	→	19
48, 49	→	20
46, 47	→	21
44, 45	→	22
42, 43	→	23
58, 59	→	24
40, 41	→	25
38, 39	→	26
36, 37	→	27
32, 33	→	28
34, 35	→	29

G20-12→R12

Vertices of G20-12		Vertices of R12
0, 1, 11, 12, 13, 14	→	0
2, 3, 15, 16, 17, 18	→	1
4, 5, 19, 20, 21, 22	→	2
6, 7, 23, 24, 25, 26	→	3
8, 9, 10, 27, 28, 29	→	4
30, 31, 32, 59, 60, 61	→	5
33, 34, 62, 63, 64, 65	→	6
105, 106, 107, 108, 118, 119	→	7
90, 91, 92, 109, 110, 111	→	8
57, 58, 86, 87, 88, 89	→	9
93, 94, 95, 96, 112, 113	→	10
97, 98, 99, 100, 114, 115	→	11
101, 102, 103, 104, 116, 117	→	12
39, 40, 68, 69, 70, 71	→	13
35, 36, 37, 38, 66, 67	→	14
41, 42, 43, 44, 72, 73	→	15
45, 46, 74, 75, 76, 77	→	16
47, 48, 49, 50, 78, 79	→	17
51, 52, 80, 81, 82, 83	→	18
53, 54, 55, 56, 84, 85	→	19

K20-12→R20

Vertices of K20-12		Vertices of R20
0, 1, 2, 3, 4	→	0
11, 12, 25, 26, 27	→	1
42, 43, 44, 45, 54	→	2
30, 31, 32, 46, 47	→	3
5, 6, 16, 17, 18	→	4
7, 8, 19, 20, 21	→	5
33, 34, 35, 48, 49	→	6
55, 56, 57, 58, 59	→	7
39, 40, 41, 52, 53	→	8
9, 10, 22, 23, 24	→	9
36, 37, 38, 50, 51	→	10
13, 14, 15, 28, 29	→	11

G6-8→K6-8

Vertices of G6-8		Vertices of K6-8
0, 1	→	0
18, 19	→	1
20, 21	→	2
46, 47	→	3
44, 45	→	4
22, 23	→	5
32, 33	→	6
42, 43	→	7
40, 41	→	8
34, 35	→	9
36, 37	→	10
38, 39	→	11
12, 13	→	12
14, 15	→	13
16, 17	→	14
2, 3	→	15
4, 5	→	16
6, 7	→	17
8, 9	→	18
10, 11	→	19
26, 27	→	20
24, 25	→	21
30, 31	→	22
28, 29	→	23

K6-8→R6

Vertices of K6-8		Vertices of R6
0, 15, 16	→	0
3, 4, 17	→	1
2, 5, 21	→	2
1, 14, 20	→	3
11, 12, 19	→	4
7, 8, 18	→	5
6, 9, 22	→	6
10, 13, 23	→	7

G6-8→T6

Vertices of G6-8		Vertices of T6
0, 5	→	0
6, 47	→	1
45, 46	→	2
21, 22	→	3
20, 25	→	4
19, 26	→	5
17, 18	→	6
1, 2	→	7
3, 4	→	8
11, 12	→	9
13, 38	→	10
14, 37	→	11
15, 28	→	12
16, 27	→	13
29, 36	→	14
30, 35	→	15
31, 32	→	16
23, 24	→	17
33, 34	→	18
41, 42	→	19
8, 43	→	20
7, 44	→	21
9, 40	→	22
10, 39	→	23

K6-8→R8

Vertices of K6-8		Vertices of R8
0, 1, 2, 3	→	0
12, 13, 14, 15	→	1
20, 21, 22, 23	→	2
4, 5, 6, 7	→	3
8, 9, 10, 11	→	4
16, 17, 18, 19	→	5

G20-12→K20-12

Vertices of G20-12		Vertices of K20-12
0, 1	→	0
2, 3	→	1
4, 5	→	2
6, 7	→	3
8, 9	→	4
10, 29	→	5
11, 12	→	6
13, 14	→	7
15, 16	→	8
17, 18	→	9
19, 20	→	10
21, 22	→	11
23, 24	→	12
25, 26	→	13
27, 28	→	14
31, 32	→	15
30, 59	→	16
57, 58	→	17
55, 56	→	18
53, 54	→	19
51, 52	→	20
49, 50	→	21
47, 48	→	22
45, 46	→	23
43, 44	→	24
41, 42	→	25
39, 40	→	26
37, 38	→	27
35, 36	→	28
33, 34	→	29
62, 63	→	30
60, 61	→	31
88, 89	→	32
86, 87	→	33
84, 85	→	34
82, 83	→	35
80, 81	→	36
78, 79	→	37
76, 77	→	38
74, 75	→	39
72, 73	→	40
70, 71	→	41
68, 69	→	42
66, 67	→	43
64, 65	→	44
105, 106	→	45
107, 108	→	46
90, 109	→	47
91, 92	→	48
93, 94	→	49
95, 96	→	50
97, 98	→	51
99, 100	→	52
101, 102	→	53
103, 104	→	54
116, 117	→	55
118, 119	→	56
110, 111	→	57
112, 113	→	58
114, 115	→	59

G20-12→T20

Vertices of G20-12		Vertices of T20
1, 2	→	0
0, 9	→	1
10, 11	→	2
12, 55	→	3
13, 54	→	4
14, 15	→	5
16, 49	→	6
17, 48	→	7
18, 19	→	8
3, 4	→	9
5, 6	→	10
7, 8	→	11
26, 27	→	12
28, 31	→	13
29, 30	→	14
58, 59	→	15
56, 57	→	16
85, 86	→	17
83, 84	→	18
52, 53	→	19
50, 51	→	20
79, 80	→	21
77, 78	→	22
46, 47	→	23
44, 45	→	24
20, 43	→	25
21, 42	→	26
22, 23	→	27
24, 37	→	28
25, 36	→	29
34, 35	→	30
32, 33	→	31
61, 62	→	32
60, 89	→	33
88, 90	→	34
87, 91	→	35
92, 93	→	36
82, 94	→	37
81, 95	→	38
96, 97	→	39
76, 98	→	40
75, 99	→	41
73, 74	→	42
71, 72	→	43
40, 41	→	44
38, 39	→	45
67, 68	→	46
65, 66	→	47
64, 106	→	48
63, 107	→	49
108, 109	→	50
110, 119	→	51
111, 112	→	52
113, 114	→	53
115, 116	→	54
100, 101	→	55
70, 102	→	56
69, 103	→	57
104, 105	→	58
117, 118	→	59

G20-12→T12

Vertices of G20-12		Vertices of T12
1, 14	→	0
2, 15	→	1
3, 18	→	2
4, 19	→	3
5, 22	→	4
6, 23	→	5
7, 26	→	6
8, 27	→	7
9, 10	→	8
0, 11	→	9
12, 13	→	10
54, 55	→	11
56, 85	→	12
57, 86	→	13
58, 89	→	14
59, 60	→	15
30, 31	→	16
28, 29	→	17
32, 61	→	18
33, 62	→	19
34, 65	→	20
35, 66	→	21
36, 37	→	22
24, 25	→	23
38, 67	→	24
39, 68	→	25
40, 71	→	26
41, 72	→	27
42, 43	→	28
20, 21	→	29
44, 73	→	30
45, 74	→	31
46, 77	→	32
47, 78	→	33
48, 49	→	34
16, 17	→	35
50, 79	→	36
51, 80	→	37
52, 83	→	38
53, 84	→	39
81, 82	→	40
94, 95	→	41
93, 112	→	42
92, 111	→	43
90, 91	→	44
87, 88	→	45
109, 110	→	46
108, 119	→	47
106, 107	→	48
63, 64	→	49
105, 118	→	50
104, 117	→	51
102, 103	→	52
69, 70	→	53
101, 116	→	54
100, 115	→	55
98, 99	→	56
75, 76	→	57
97, 114	→	58
96, 113	→	59

References

[1] Hitotsumatsu, S. (2002) *Solve the polyhedra*. Tokai University Press, 2002 (in Japanese).

[2] *WolframMathWorld* http://mathworld.wolfram.com/topics/ArchimedeanSolids.html

[3] Kametaka, Y., Yamagishi, H., Watanabe, K., Nagai, A. and Takemura, K. (2007) Riemann zeta function, Bernoulli polynomials and the best constant of Sobolev inequality. *Sci. Math. Japan* **65**, 333-359.

[4] Kametaka, Y., Watanabe, K. and Nagai, A. (2005) The best constant of Sobolev inequality in an *n* dimensional Euclidean space. *Proc. Japan Acad.* **81**, *Ser. A,* 57-60.

[5] Kametaka, Y., Watanabe, K., Nagai, A. and Pyatkov, S. (2005) The best constant of Sobolev inequality in an *n* dimensional Euclidean space. *Sci. Math. Japan* **61**, 15-23.

[6] Yamagishi, H., Kametaka, Y., Nagai, A., Watanabe, K. and Takemura, K. (2009) Riemann zeta function and the best constants of five series of Sobolev inequalities. *RIMS Kôkyûroku Bessatsu* **B13**, 125-139.

[7] Yamagishi, H., Kametaka, Y., Takemura, K., Watanabe, K. and Nagai, A. (2009) The best constant of discrete Sobolev inequality corresponding to a bending problem of a beam under tension on an elastic foundation. *Trans. Japan Soc. Ind.*

Appl. Math. **19**, 489-518 (in Japanese).

[8] Kametaka, Y., Nagai, A., Yamagishi, H., Takemura, K. and Watanabe, K. (2014) The best constant of dicrete Sobolev inequality on the C60 fullerene buckyball. *arXiv.org e-Print archive.* http://arxiv.org/abs/1412.1236

[9] Kametaka, Y., Watanabe, K., Yamagishi, H., Nagai, A. and Takemura, K. (2011) The best constant of discrete Sobolev inequality on regular polyhedron. *Trans. Japan Soc. Ind. Appl. Math.* **21**, 289-308 (in Japanese).

[10] Nagai, A., Kametaka, Y., Yamagishi, H., Takemura, K. and Watanabe, K. (2008) Discrete Bernoulli polynomials and the best constant of dicrete Sobolev inequality. *Funkcial. Ekvac.* **51**, 307-327.

[11] Yamagishi, H., Kametaka, Y., Nagai, A., Watanabe, K. and Takemura, K. (2014) The best constant of dicrete Sobolev inequality on truncated polyhedra. Abstract of *the 10th meeting of the union of research activity groups, Japan SIAM,* Kyoto Univ., Kyoto, 9-10 March 2014 (in Japanese).
http://chaosken.amp.i.kyoto-u.ac.jp/_src/sc2490/jsiam_s2_no4_yamagisi_abst.pdf

[12] Yamagishi, H., Kametaka, Y., Nagai, A., Watanabe, K. and Takemura, K. (2013) The best constant of three kinds of dicrete Sobolev inequalities on regular polyhedron. *Tokyo J. Math.* **36**, 253-268.

[13] Kroto, H.W., Heath, J.R., O'Brien, S.C., Curl, R.F. and Smalley, R.E. (1985) C60: Buckminsterfullerene. *Nature* **318**, 162-163.

[14] Brezis, H. (1983) *Analyse fonctionnelle: théorie et applications.* Masson, Paris.

[15] Talenti, G. (1976) The best constant of Sobolev inequality. *Ann. Mat. Pura Appl.* **110**, 353-372.

[16] Marti, J.T. (1983) Evaluation of the least constant in Sobolev's inequality for $H^1(0, s)$ *SIAM J. Numer. Anal.* **20**, 1239-1242.

[17] Shinohara, H. and Saito, Y. (2011) *Science of fullerenes and CNTs.* Nagoya University Press, Nagoya.

[18] Hawkins, J.M., Meyer, A.L., Timothy, A., Loren, S. and Hollander, F.J. (1991) *Science* **252**, 312.

[19] Hedberg et al. (1991) *Scinence* **254**, 410.

[20] Yannoni, C.S., Bernier, P.P., Bethune, D.S., Meijer, G. and Salem, J.R. (1991) *J. Am. Chem. Soc.* **113**, 3190.

[21] Sasaki, Y. (2015) The best constant of discrete Sobolev inequality on a weighted truncated tetrahedron, *World J. Engi. Tech.* **3**, 149-154.