A Topological Condition for a Linear Time-Varying Electric Circuit to be Represented by a Canonical Equation

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Abstract

As is well known, electric circuits consisting of a finite number of resistors, capacitors, and inductors including mutual inductances can be described by the Lagrange-Maxwell equations, which are equivalent to loop equations due to Kirchhoff. However, if we further intend state-space approach to the representation of electric circuits, there arises a problem of finding conditions under which such an electric circuit may be represented by a canonical equation, i.e., by a state equation. It is decisive that electric circuits are distinctive from the Newtonian mechanical systems in the point that the quadratic form which represents magnetic energy stored in all the inductors, which corresponds to the kinetic energy of a mechanical system, is not necessarily positive definite. This paper presents, in the necessary and sufficient manner, a topological condition for an electric circuit prescribed by a Lagrange-Maxwell equation to be represented by a state equation, without any modification such as insertion of any excess inductors.

Keywords: State equation of electric circuits, topological condition for state-space representation

1. Introduction

It is well known that an electric circuit consisting of a finite number of resistors, capacitors, and inductors including mutual inductances can be described by a Lagrange-Maxwell's equation, which is equivalent to a loop equation due to Kirchhoff. If the rank of the coefficient matrix of inductors of the loop equation is equal to the number of the fundamental loops of the circuit, then the loop equation can be rewritten by a canonical equation with respect to the state variables, i.e., to electric charges and magnetic fluxes stored, respectively, in all the capacitors and the inductors in the circuit under consideration.

While, if the rank of the coefficient matrix of inductors is less than the number of the fundamental loops, then the loop equation of the circuit can not necessarily be represented by a canonical equation. Therefore, if we intend to rewrite the loop equation in a canonical form, that is, if we take state-space approach to the circuit representation, then we have to clarify conditions under which the electric circuit be representable by a canonical equation. It is noted that electric circuits are distinctive from the Newtonian mechanical systems in the point that the quadratic form with respect to the magnetic fluxes corresponding to the kinetic energy in the mechanical systems is not necessarily positive definite. There have so far been some contributions in which such conditions as mentioned above are given only in a sufficient manner or only as algebraic conditions[1].

In this paper, provided that an electric circuit can be described by a Lagrange-Maxwell equation, we clarify a topological condition, in the necessary and sufficient manner, for the electric circuit to be represented by a state

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equation without modification such as insertion of any excess inductors.

2. Circuit Equations

Let $G(V, E)$ be the connected and oriented graph of a linear time-varying circuit N, where V and E are respectively sets of all vertices (nodes) and branches contained in G. Let $E = E_L \cup E_C \cup E_R$, where

$$\begin{aligned} E_L &= \{b_\lambda;\ \lambda = 1, 2, \ldots, n_L\} \\ E_C &= \{b_\gamma;\ \gamma = 1, 2, \ldots, n_C\} \\ E_R &= \{b_\rho;\ \rho = 1, 2, \ldots, n_R\} \end{aligned} \tag{1}$$

are respectively sets of branches each of which contains an inductor (L), a capacitor (C), and a resistor (R), in which Greek letters λ, γ, and ρ are respectively used for discriminating L, C, R from each other. It is noted that E_L, E_C, and E_R may have common branches, and that electric sources are only voltage sources contained in branches.

We shall hereafter assume that L, C, and R are positive valued so that

$$0 < L_\lambda(t) < L_\lambda, \quad 0 < C_\gamma(t) < C_\gamma, \quad 0 < R_\rho(t) < R_\rho, \tag{2}$$

and may smoothly vary in time, that is, their differentials with respect to time are continuous. Each branch current i is represented by the derivative of electric charge q as

$$i = \dot{q} = \frac{dq}{dt}. \tag{3}$$

Next, let E be partitioned into a tree

$$E_{tree} = \{t_i;\ i = 1, 2, \ldots, n-1\}, \quad n = |V| \tag{4a}$$

and cotree

$$E_{cotree} = \{t_k^c;\ k = 1, 2, \ldots, l\}, \quad l = |E| - |V| + 1 \tag{4b}$$

and let

$$C = \begin{bmatrix} -M^T \\ I_{l\times l} \end{bmatrix}, \quad \text{where } M = [m_{i\times k}] \tag{5}$$

be the fundamental cut-set matrix of $G(V, E)$, where $I_{l\times l}$ is the $l \times l$ identity matrix, $m_{i\times k} \in \{0, 1, -1\}$ in the $l \times (n-1)$ submatrix M and its rows correspond to branches of cotree E_{cotree} and columns to those of the tree E_{tree}. Then, all the branch currents are expressed by the l-vector

$$\dot{\boldsymbol{q}} = (\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_l) \tag{6}$$

$\dot{q}_k$'s being the cotree currents, multiplied by the fundamental loop matrix

$$R = [M, I_{l\times l}] \tag{7}$$

derived from (5).

To describe the circuit equation, following Lagrange-Maxwell, we define three kinds of energy functions

$$T = T(\dot{q}_\lambda, t) = \frac{1}{2}\sum_{\lambda=1}^{n_L} L_\lambda(t)\,\dot{q}_\lambda^2, \tag{8a}$$

$$U = U(q_\gamma, t) = \frac{1}{2}\sum_{\gamma=1}^{n_C} C_\gamma(t)\,q_\gamma^2, \tag{8b}$$

$$F = F(\dot{q}_\rho, t) = \frac{1}{2}\sum_{\rho=1}^{n_R} R_\rho(t)\,\dot{q}_\rho^2, \tag{8c}$$

which denote respectively magnetic energy stored in all the inductors, electro static energy stored in all the capacitors, and dissipation energy. Then, substituting the cotree variables $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$ into q_γ's, $\dot{q}_\lambda$'s, $\dot{q}_\rho$'s, we obtain the circuit equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) + \frac{\partial U}{\partial q_k} + \frac{\partial F}{\partial \dot{q}_k} = e_k, \quad k = 1, 2, \ldots, l, \tag{9}$$

the e_k's being e.m.f.'s summed up along k-th loop; this is an alternative expression of the well-known loop equations.

3. Derivation of the state equation

Following the Hamilton-Jacobi formalism, we can choose as two kinds of state variables, i.e., generalized coordinates and momentums respectively, the electric charges stored in capacitors and the magnetic fluxes interlinked inductors in the circuit under consideration. In our case, the generalized coordinates are defined by (6), and the generalized momentums are defined by

$$p_k=\frac{\partial T(\dot{q}_1, \dots, \dot{q}_f)}{\partial \dot{q}_k}, \quad k=1, 2, \dots, f, \tag{10}$$

where $\dot{q}_1, \cdots, \dot{q}_f$ are assigned to respective currents in the cotree:

$$\{t_k^C;\ k=1, 2, \dots, f\}\subset E_{cotree}, \tag{11}$$

which are necessary for representing all the branches of E_L. We shall assume that $T(\dot{q}_1, \dots, \dot{q}_f)$ is positive definite with respect to $\dot{q}_1, \dots, \dot{q}_f$. Then, $\dot{q}_1, \dots, \dot{q}_f$ can be determined uniquely from (10) as

$$\dot{q}_k=g_{1k}(p_1, \dots, p_f), \quad k=1, 2, \dots, f. \tag{12}$$

It is obvious from (10) that

$$p_k=0, \quad k=f+1, f+2, \dots, l, \tag{13}$$

i.e.,

$$p_k=\frac{\partial T(\dot{q}_1, \dots, \dot{q}_f)}{\partial \dot{q}_k}=0, \quad f+1\le k\le l \tag{14}$$

hold. Therefore, substituting (10) and (14) in (9), we have

$$\dot{p}_k=-\frac{\partial U}{\partial q_k}-\frac{\partial F}{\partial \dot{q}_k}+e_k, \quad 1\le k\le f, \tag{15a}$$

$$0=-\frac{\partial U}{\partial q_k}-\frac{\partial F}{\partial \dot{q}_k}+e_k, \quad f+1\le k\le l. \tag{15b}$$

Next, it is easily seen from (8c) and (12) that $\frac{\partial F}{\partial \dot{q}_k}$ can be expressed as

$$\frac{\partial F}{\partial \dot{q}_k}=f_{11k}(p_1, \dots, p_f)+f_{12k}(\dot{q}_{f+1}, \dots, \dot{q}_l), \quad k=1, 2, \dots, f, \tag{16a}$$

$$\frac{\partial F}{\partial \dot{q}_k}=f_{21k}(p_1, \dots, p_f)+f_{22k}(\dot{q}_{f+1}, \dots, \dot{q}_l), \quad k=f+1, \dots, l, \tag{16b}$$

where f_{11k}, f_{12k}, f_{21k}, and f_{22k} are all of linear form, and that $f_{22k}(\dot{q}_{f+1}, \dots, \dot{q}_l)$ can be written as

$$f_{22k}=\frac{\partial F_2}{\partial \dot{q}_k}, \quad k=f+1, \dots, l, \tag{16c}$$

where $F_2=F_2(\dot{q}_{f+1}, \dots, \dot{q}_l)$ is, in general, a nonnegative quadratic form.

It is seen by substituting (16a) and (16b) in (15a) and (15b) that $\dot{q}_{f+1}, \dots, \dot{q}_l$ can be eliminated from (15a) and (15b) only if (15b) can be solved uniquely in $\dot{q}_{f+1}, \dots, \dot{q}_l$ by substituting (16b) in (15b), i.e., if $\dot{q}_k$'s can be written in general as

$$\dot{q}_k=g_{2k}(q_1, \dots, q_l, p_1, \dots, p_f), \quad f+1\le k\le l. \tag{17}$$

This is possible only if F_2 is positive definite.

From the argument above, under the assumption that both $T(\dot{q}_1, \dots, \dot{q}_f)$ and $F_2(\dot{q}_{f+1}, \dots, \dot{q}_l)$ are positive definite, we conclude that (9) can be represented by the state equation of the form:

$$\begin{bmatrix}\dot{\boldsymbol{q}}\\ \dot{\boldsymbol{p}}\end{bmatrix}=\boldsymbol{A}\begin{bmatrix}\boldsymbol{q}\\ \boldsymbol{p}\end{bmatrix}+\boldsymbol{Be}. \tag{18}$$

Remark 1: A distinct point of the description by the state equation is that it admits rather weak constraints on the continuity of elements of the matrix $\boldsymbol{A}$.

4. Topological condition for T and F_2 to be positive definite

We shall first state, from the graph theoretical viewpoint, the necessary and sufficient conditions for T and F_2 to be positive definite.

Definition 1: For a given connected graph $G(V, E)$, let $E_1 \subseteq E$ be a subset of E, where $|E_1| = p \leq b = |E|$, and denote currents of the branches belonging to E_1 by a row p-vector $\boldsymbol{u} \in R^p$. Then, the branches of E_1 and the quadratic form $\boldsymbol{u}W\boldsymbol{u}^t$ is said to be *positively weighted* if $W = \mathrm{diag}[W_1, \dots, W_p]$, where $W_k > 0$.

Definition 2: By E_{tree} and E_{cotree} denote respectively the set of branches of an arbitrarily chosen tree and its cotree out of the $G(V, E)$ given in Definition 1, and denote

$$E_1 \cap E_{tree} = \{t_k;\ k=1, \dots, \mu\}, \quad \mu \leq p \leq n-1, \tag{19}$$

for E_1 in Definition 1, in which $n = |V|$. Let $C(t_k)$ be the fundamental cut set such that it contains only t_k out of E_{tree}, and let $C'(t_k) = C(t_k) - \{t_k\} \subset E_{cotree}$ be expressed as

$$C'(t_k) = \{\gamma_{kj} t_j^C;\ j=1, \dots, \nu\}, \quad 0 < \nu \leq l, \quad (k=1, \dots, \mu), \tag{20}$$

in which $l = |E_{cotree}| = b - n + 1$, where $b = |E|$, and γ_{kj}'s are elements of a $\mu \times \nu$ submatrix of the fundamental cut set matrix of $G(V, E)$. Then, for

$$S = S(E_1) = \bigcup_{k=1}^{\mu} C'(t_k), \tag{21}$$

the sets of links

$$S_1(E_1) = S - S \cap E_1 \cap E_{cotree}, \tag{22}$$

$$S_2(E_1) = S \cap E_1 \cap E_{cotree}, \tag{23}$$

$$S_3(E_1) = E_1 \cap E_{cotree} - S \cap E_1 \cap E_{\mathrm{cotree}}, \tag{24}$$

are uniquely defined as relatively disjoint sets by the decomposition of the set $S \cup (E_1 \cap E_{\mathrm{cotree}})$, which are called the *fundamental subsets of the cotree* associated with an E_1 and E_{tree} on $G(V, E)$.

Definition 3: Let $\boldsymbol{u}^C \in R^f$ be a row f-vector made of the branch currents of the joint set $S_1(E_1) \cup S_2(E_1) \cup S_3(E_1)$, where $f = f_1 + f_2 + f_3$ with $f_k = |S_k(E_1)|$, $k = 1, 2, 3$, and write the p-vector $\boldsymbol{u} \in R^p$ in Definition 1 as $\boldsymbol{u} = \boldsymbol{u}^C M$, in which M is an $f \times p$ submatrix of the fundamental loop matrix of $G(V, E)$. Then, the quadratic form $\boldsymbol{u}^C M W M^t \boldsymbol{u}^{Ct}$ is called the *compact form of* $\boldsymbol{u}W\boldsymbol{u}^t$.

Remark 2: It is easily seen from Definition 2 and 3 that $f = |S_1(E_1) \cup S_2(E_1) \cup S_3(E_1)| = f_1 + f_2 + f_3$ and $f \leq p$ hold in general, since $S_k(E_1)$'s are relatively disjoint.

Now, we state the theorem.

Theorem 1: With symbols and notations being the same as in Definition 1-3, the compact form of a positively weighted quadratic form $\boldsymbol{u}W\boldsymbol{u}^t$ is positive definite, if and only if the fundamental subsets $S_1(E_1)$, $S_2(E_1)$, and $S_3(E_1)$ of the cotree associated with an E_1 and E_{tree} on $G(V, E)$ satisfy the conditions:

(i) If $S_1(E_1)$ is written as

$$S_1(E_1) = \bigcup_{j=1}^{f_1} \{\gamma_{ji} t_i^C;\ i=1, \dots, \nu\}, \quad f_1 = |S_1|, \tag{25}$$

where γ_{kj}'s are elements of a $\mu \times \nu$ submatrix of the fundamental cut set matrix of $G(V, E)$, then the vectors

$$\gamma_j = \{\gamma_{j1}, \gamma_{j2}, \dots, \gamma_{j\nu}\}, \quad j=1, \dots, f_1 \tag{26}$$

are linearly independent.

(ii) The vectors

$$\gamma_j = \{\gamma_{j1}, \gamma_{j2}, \dots, \gamma_{j\nu}\}, \quad j = f_1+1, f_1+2, \dots, f_1+f_2 \tag{27}$$

associated with

$$S_2(E_1)=\bigcup_{j=f_1+1}^{f_1+f_2}\{\gamma_{jk}t_k^C;\ k=1,\ \dots,\ \nu\},\quad f_2=|S_2| \tag{28}$$

are linearly dependent on the γ_j's defined in (i), but all branches belonging to $S_2(E_1)$ are positively weighted.

(iii) All branches belonging to S_3 are positively weighted.

(Proof) Given E_1 on $G(V, E)$, for arbitrarily assigned tree E_{tree}, the decomposition into the fundamental subsets of the cotree of E_{tree}, i.e., into S_1, S_2, S_3 is unique and these subsets are relatively disjoint. Therefore, in order for the compact form $\boldsymbol{u}^C MWM^t \boldsymbol{u}^{Ct}$ of $\boldsymbol{u}W\boldsymbol{u}^t$ to be positive definite, it is necessary and sufficient that the matrix MWM^t is of full rank, and this is obviously satisfied under the condition (i), (ii), and (iii). Q.E.D.

To sum up the above-mentioned statement and the argument in Section 3, the next theorem is obtained as a corollary to Theorem 1.

Theorem 2: The necessary and sufficient condition for a time-varying LCR circuit N to be described by a state equation in terms of charges $\boldsymbol{q}=(q_1, q_2, \dots, q_l)$ on respective links of the cotree and generalized magnetic flux $\boldsymbol{p}=(p_1, p_2, \dots, p_l)$, which is determined from the magnetic energy $T(\dot{\boldsymbol{q}}, t)$ with $\dot{\boldsymbol{q}}=(\dot{q}_1, \dot{q}_2, \dots, \dot{q}_l)$, is that there exists a tree E_{tree} on the connected graph $G(V, E)$ on N such that if each of E_L and

$$E'_R=E_R-E_R\cap(S(E_L)\cup(E_L\cap E_{cotree})) \tag{29}$$

is assumed to be the set E_1 of Theorem 1, then Theorem 1 holds respectively in each case for $E_1=E_L$ and $E_1=E'_R$.

5. Concluding Remarks

We have obtained in this paper a topological condition, under which a linear electric circuit, time-invariant and/or time-varying, can be described by a state equation. Further, state-space approach to nonlinear electric circuits will be of importance especially in studying chaotic behavior of the electric circuits. It is seen from the discussion in Section 3 that the argument on the elimination of the time-differentials of generalized coordinates, $\dot{q}_{f+1}, \dots, \dot{q}_l$, shown with regard to the equations (10) through (15b), can be generalized to some nonlinear circuits consisting of passive linear inductors, passive capacitors, linear and/or nonlinear, and nonlinear resistors by applying the implicit function theorem in place of the above-mentioned argument on the elimination of $\dot{q}_{f+1}, \dots, \dot{q}_l$. Therefore, the topological condition obtained in this paper will be valid in some extent in the nonlinear circuits.

Reference

[1] R.A. Rohrer, CIRCUIT THEORY, An Introduction To The State Variable Approach, McGraw-Hill Kogakusha, Ltd., 1970.