On the Fourier Analysis and the Lagrange Interpolation from the Viewpoint of Signal Theory

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ABSTRACT

The Fourier analysis is of importance not only as just one of the mathematical tools for processing signals and noise, but also for its intrinsic role in the representation of almost every informational feature of physical objects. This paper discusses some aspects related to the Fourier analysis, sampling theory, and the Lagrange interpolation from the viewpoint of signal theory. First, the intrinsic role of the Fourier analysis is clarified in relation to the wave equations and the human sensation in vision and hearing. Next, an algebraic approach to the Fourier analysis based on the Lagrange interpolation is shown as the theoretical basis of the sampling technique. The equivalence between the discrete Fourier transform (DFT) and the Lagrange interpolation on the unit circle in the complex Z-plane, on which the so-called Z-transform is defined, is shown in an algebraic way. The Fourier series expansion is derived as a limit of a Lagrange interpolation polynomial.

1. Introduction——Role of the Fourier Analysis in the Signal Theory

In preparation of a monograph on a review of the development of signal processings in this century (1), contents of this paper has been written, as a byproduct, on the fundamental role of the Fourier analysis via the Lagrange interpolation as the basis of the signal theory.

It is our basic knowlage that every visual information exibited out of the physical objects is carried by light, or by electromagnetic wave, and similarly every audio information in the speach, music, etc. is carried by sound wave, as physical signals to the receptors of our sense organs and the recognition of their features or states are made respectively (2).

Distinguishable point common in both the cases mentioned above is that those waves are solutions of respective wave equations of the same form, that is, they are necessarily expressible in D'Alembert's general solutions, which are represented in terms of arbitrary functions; for example, in one-dimensional case, the general solution of the wave equation described as

$$\frac{\partial^2 \varphi}{\partial x^2}-\frac{1}{U^2}\frac{\partial^2 \varphi}{\partial t^2}=0, \tag{1. 1}$$

U denoting the wave velocity, can be written as

$$\varphi(x, t)=f(x-Ut)+g(x+Ut), \tag{1. 2}$$

where f and g are arbitrary functions which will be precisely defined later. On the one hand, if $\varphi(x, t)$ is subject to a boundary condition such that $\varphi(x, t)$ vanish always at two boundary points, say $x=0$ and $=a$, then $f(x)$ is arbitrarily given as a general solution besides satisfying $f(0)=f(a)=0$, and also shown to be expressed, as an alternative general solution, in a series of sine functions with the period of 2a; that is, $\varphi(x, t)$ must be expressible as a Fourier sine series in the sense of the point function. This fact can be extended, in general, to aperiodic cases provided that f and g in (1.2) belong to $L_1(-\infty, +\infty)$ with respect to t; that is, $\varphi(x, t)$ can be represented as a point

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function by means of the Fourier analysis (3), (4).

Moreover, though it is beyond our sensation to recognize the existence of the photon, if this concept can be interpreted by a wave-packet, and if the wave-packet is to be a solution of the wave equation, then this solution must not be a point function but a set function or a generalized function (5). In other words, the class of general solutions of the wave equation may be much wider than the class of conventional point functions.

Mathematical verification of the statement above was established by Dirichlet, Riemann, Lebesgue, Shwartz, and others. As is well known, Dirichlet first clarified a sufficient condition that a periodic function $f(x)$ can be represented by a Fourier series if $f(x)$ is sectionally smooth (6), and finally Riemann and Lebesgue gave the necessary and sufficient condition for an $f(x)$ as a point function to be expressible as a Fourier series that $f(t)$ is periodic and of bounded variation in the vicinity of any instant t under consideration (4), (5).

Referring to the above-mentioned argument, and considering that any signal must be stochastic, i.e., a collection of such signals must have Shannon's positive entropy, we may conclude that the signals available in any field, especially in the electrical, electronic, and information engineering can be specified by the condition that a signal $f(t)$ belongs to $L_1(-\infty, +\infty)$ and is of bounded variation in the vicinity of any instant t under consideration, so that Dirichlet's "sectionally smooth" condition suffices it.

We shall clarify in the following sections basic properties related to the sampling and interpolation of the signals specified above in relation to the Fourier transform.

2. The Lagrange Polynomial Interpolation

As the basis for the following discussion, we shall first refer to the well-known Lagrange formula of polynomial interpolation (7).

Theorem 1: Given n + 1 points $x=x_1, x_2, x_3, \ldots, x_{n+1}$ on the real x-axis, and correspondingly real values $y=y_1, y_2, y_3, \ldots, y_{n+1}$, then a polynomial $f(x)$ of degree n satisfying

$$f(x_k)=y_k,\ k=1, 2, \ldots, n+1 \tag{2. 1}$$

is uniquely determined by

$$f(x)=\sum_{k=1}^{n+1} y_k \frac{\Phi(x)}{(x-x_k)\Phi'(x_k)}, \tag{2. 2a}$$

in which

$$\Phi(x)=(x-x_1)(x-x_2)(x-x_3)\ldots(x-x_{(n+1)}) = \prod_{k=1}^{n+1}(x-x_k). \tag{2. 2b}$$

Next, the polynomial interpolation on the unit circle in the complex plane can be derived directly from Theorem 1, by rewriting x_k and $y_k (k=1, 2, \ldots, n+1)$ in the formula (2. 2a) and (2. 2b) respectively by

$$W_k=e^{jk\frac{2\pi}{n+1}} \quad (k=1, 2, \ldots, n+1)$$

and

$$f_k=f(W_k), \quad k=1, 2, \ldots, n+1.$$

That is, we obtain the next theorem.

Theorem 2 : Given $(n+1)$ points $W_k=e^{jk\frac{2\pi}{n+1}}$ $(k=1, 2, \ldots, n+1)$ on the unit circle in the complex plane, then a polynomial $f(W)$ of degree n satisfying

$$f_k=f(W_k), \quad k=1, 2, \ldots, n+1 \tag{2. 3a}$$

is uniquely determined by

$$f(W)=\sum_{k=1}^{n+1} f_k \frac{\Phi(W)}{(W-W_k)\Phi'(W_k)}, \tag{2. 3b}$$

in which

$$\Phi(W)=(W-W_1)(W-W_2)...(W-W_n)(W-W_{n+1})$$
$$=\prod_{k=1}^{n+1}(W-e^{jk\frac{2\pi}{n+1}})=W^{n+1}-1. \quad (2.3c)$$

3. Sampling Theorem as an Extension of the Lagrange Polynomial Interpolation

We shall use a conventional terminology, an analog signal or analog signals, instead of time-series of the signal in the continuous time. Our sense organs, visual and hearing, have limited resolving power, and hence, the frequency response of our sensation shows band-limited characteristics. This fact will be understood from our experiences such that we can enjoy listening to a violine sonata by a CD player and watching beautiful scenary in TV, of which all the signals are composed of discrete pulse trains. The principle by which analog signals can be equivalently represented by discrete time series is summalized in the following

Sampling Theorem (8), (9) : Let the Fourier transform of an analog signal $f(t)\in L_1(-\infty, +\infty)$ be

$$F(j\omega)=\int_{-\infty}^{+\infty}f(t)e^{-j\omega t}dt, \quad (3.1)$$

and let $f(t)$ satisfy the condition of band-limitation:

$$|F(j\omega)|=0 \qquad |\omega|>\omega_C. \quad (3.2)$$

Then, $f(t)$ can be represented in terms of its sampled values

$$\{f(nT);\quad n=0, \pm1, \pm2, ...\}, \quad (3.3a)$$

in which

$$T=\frac{\pi}{\omega_C}=\frac{2\pi}{W} \qquad (W=2\omega_C), \quad (3.3b)$$

as

$$f(t)=\sum_{n=-\infty}^{+\infty}f(nT)Sa(t-nT) \quad (3.4a)$$

where $Sa(t)$, the sampling function, is defined by

$$Sa(t)=\frac{\sin(\omega_C t)}{\omega_C t}. \quad (3.4b)$$

(Proof) The formulas (3.4a) and (3.4b) are considered to be an extension of the Lagrange Polynomial Interpolation. From this view-point, an alternate proof is given as follows.

If we substitute

$$x_k \quad (k=0, \pm1, \pm2, ..., \pm(n+1)) \quad (3.5a)$$

for x_k in the formula of the Lagrange Polynomial Interpolation, then we obtain

$$f(x)=\sum_{k=-(n+1)}^{n+1}y_k\frac{\Phi(x)}{(\frac{x}{x_k}-1)\Phi'(x_k)}, \quad (3.5b)$$

in which

$$\Phi(x)=\prod_{k=-(n+1)}^{n+1}(\frac{x}{x_k}-1)=x\prod_{k=1}^{n+1}\left[\left(\frac{x}{x_k}\right)^2-1\right]. \quad (3.5c)$$

If we put $x=\frac{t}{T}$ in (3.5c), then it can be rewritten as

$$\Phi(\frac{t}{T})=\frac{t}{T}\prod_{k=-(n+1)}^{n+1}\left[\left(\frac{t}{kT}\right)^2-1\right] \quad (3.6)$$

Next, with reference to the well-known expansion of the sine function as an infinite product (3)

$$\sin\pi x=\pi x(1-\frac{x^2}{1^2})(1-\frac{x^2}{2^2})(1-\frac{x^2}{3^2})...,$$

taking the limit of (3.6) as $n \to \infty$, we obtain

$$\Phi(\frac{t}{T}) = \frac{1}{\pi}\sin(\frac{\pi x}{T})$$

Therefore, considering that

$$\Phi(\frac{t}{T}) = \frac{1}{\pi}\sin(\frac{\pi x}{T}) = (-1)^k \frac{1}{\pi}\sin(\frac{\pi(t-kT)}{T}),$$

$$(\Phi'(x))_{x=x_k} = (\Phi'(\frac{t}{T}))_{t=kT} = \frac{\cos(k\pi)}{T} = \frac{(-1)^k}{T}.$$

and rewriting $f(x) = f(\frac{t}{T})$ as $f(t)$ and $y_k = f(kT)$, we have

$$f(t) = \sum_{n=-\infty}^{+\infty} f(kT)\frac{\sin\omega_C(t-kT)}{\omega_C(t-kT)}. \qquad \omega_C = \frac{\pi}{T}.$$

This is what we have wanted to prove. Q.E.D.

4. Eqivalence of the Inverse Discrete Fourier Transform and the Lagrange Polynomial Interpolation

A periodic function $f(x)$ with the period 2π of bounded variation can be extended, as is well known, in a complex Fourier series:

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{jnx}, \tag{4. 1a}$$

in which

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-jnx}dx \quad (k=0, \pm 1, \pm 2, \ldots). \tag{4. 1b}$$

In digital computation of (4. 1a) and (4. 1b), (4. 1b) is approximated by the Discrete Fourier Transform, DFT, defined in general by

$$C_n = \frac{1}{N}\sum_{k=0}^{N-1} f_k e^{-jnk\frac{2\pi}{N}} \qquad (n=0, 1, 2, \ldots, N-1), \tag{4. 2a}$$

where

$$f_k = f(x_k) \qquad (k=0, 1, 2, \ldots, N-1), \tag{4. 2b}$$

in which

$$x_k = k\frac{2\pi}{N} \qquad (k=0, 1, 2, \ldots, N-1). \tag{4. 2c}$$

For the coefficients C_n's defined by (4. 2a), the Fourier series (4. 1a) is approximated by a finite Fourier series:

$$f(x) = \sum_{n=0}^{N-1} C_n e^{jnx}. \tag{4. 2d}$$

We shall prove that the approximate Fourier series obtained as the inverse transform of the DFT defined above is equivalent to a Lagrange interpolation polynomial defined in Theorem 2.

Theorem 3 : Let $f(x)$ be a periodic function with the period 2π, and let $f(x)$ be of bounded variation. Let the DFT of $f(x)$ be

$$C_n = \frac{1}{N}\sum_{k=0}^{N-1} f_k e^{-jnk\frac{2\pi}{N}} \qquad (n=0, 1, 2, \ldots, N-1), \tag{4. 3a}$$

in which

$$\{f_k = f(x_k);\ x_k = k\frac{2\pi}{N}, \quad k=0, 1, 2, \ldots, N-1\}. \tag{4. 3b}$$

Then, the inverse DFT of (4.3a) can be expressed as

$$\sum_{k=0}^{N-1} C_n e^{jnx} = \sum_{k=0}^{N-1} f_k \frac{\Phi(W)}{(W-W_k)\Phi'(W_k)}. \tag{4. 4a}$$

where $W=e^{jx}$, and

$$W_k=e^{jk\frac{2\pi}{N}} \qquad (k=0, 1, 2, \ldots, N-1), \tag{4. 4b}$$

and

$$\begin{aligned}\Phi(W) &= (W-W_0)(W-W_1)(W-W_2)\ldots(W-W_{(N-1)})\\ &=\prod_{k=0}^{N-1}(W-e^{jk\frac{2\pi}{N}})=W^N-1.\end{aligned} \tag{4. 4c}$$

(Proof) From (4. 4c), we obtain

$$\Phi'(W_k)=Ne^{jk\frac{2\pi(N-1)}{N}}=Ne^{-jk\frac{2\pi}{N}} \qquad (k=0, 1, 2, \ldots, N-1), \tag{4. 5}$$

and hence, the right-hand side of (4. 4a) can be written as

$$F(W)=\sum_{k=0}^{N-1}\frac{f_k}{N}e^{jk\frac{2\pi}{N}}\frac{W^N-1}{(W-W_k)}. \tag{4. 6}$$

We shall rewrite (4. 6) as a polynomial in W. To do this, we consider the relation, for $0\leq k\leq N-1$,

$$\frac{W^N-1}{(W-W_k)}=\frac{W^N-W_k^N}{(W-W_k)}.$$

and the formula which is well known in elementary algebra:

$$\frac{W^N-W_k^N}{(W-W_k)}=\sum_{n=0}^{N-1}W_k^{-(n+1)}W^n. \tag{4. 7}$$

Substituting (4. 7) in (4. 6), it is rewritten as

$$\begin{aligned}F(W)&=\sum_{k=0}^{N-1}\frac{f_k}{N}e^{jk\frac{2\pi}{N}}\sum_{n=0}^{N-1}W_k^{-(n+1)}W^n\\ &=\sum_{k=0}^{N-1}\sum_{n=0}^{N-1}\frac{f_k}{N}e^{-jnk\frac{2\pi}{N}}W^n\\ &=\sum_{n=0}^{N-1}\left(\frac{1}{N}\sum_{k=0}^{N-1}f_ke^{-jnk\frac{2\pi}{N}}\right)W^n.\end{aligned} \tag{4. 8}$$

Referring to (4. 3a), (4. 8) can be expressed as

$$F(W)=\sum_{n=0}^{N-1}C_nW^n.$$

We have thus obtained Theorem 3. Q.E.D.

Corollary : It is seen from (4. 8) that, in order for

$$f_k=\sum_{n=0}^{N-1}\left(\frac{1}{N}\sum_{m=0}^{N-1}f_me^{-jnk\frac{2\pi}{N}}\right)e^{jnk\frac{2\pi}{N}} \tag{4. 9a}$$

to hold for arbitrarily given values f_k's,

$$\frac{1}{N}\sum_{n=0}^{N-1}e^{-jnm\frac{2\pi}{N}}e^{jnk\frac{2\pi}{N}}=\delta_{mk} \tag{4. 9b}$$

must hold, in which the symbol δ_{mk} denotes Kronecker's symbol defined by

$$\delta_{mk}=1\ (m=k),\ =0\ (m\neq k). \tag{4. 9c}$$

5. Fourier Transform as a Limit of the Lagrange Polynomial Interpolation

Let $f(x)$ be a periodic function with the period 2π, and denote sampled values of $f(x)$ by

$$f_k=f(x_k) \text{ for } x_k=k\frac{2\pi}{2N} \qquad (-N\leq k\leq N) \tag{5. 1}$$

Then, denoting $f(x)$ by

$$f(x)=F(W)=F(e^{jx}), \tag{5. 2a}$$

we see from Theorem 2 that $W^NF(W)$ can be expressed by a Lagrange interpolation polynomial $W^NF_N(W)$ of degree $2N$ as

$$W^N F_N(W)=\sum_{k=-N}^{N} f_k \frac{\Phi(W)}{(W-W_k)\Phi'(W_k)}, \qquad (5.2b)$$

in which

$$\Phi(W)=\prod_{k=-N}^{N}(W-e^{jk\frac{2\pi}{2N}})=W^{2N+1}-1, \qquad (5.2c)$$

and

$$\Phi'(W_k)=(2N+1)W_k^{2N}=(2N+1)W_k^{-1}. \qquad (5.2c)$$

By Eqs. (5. 2b) - (5. 2c) and considering the identities

$$\Phi(W)=W^{2N+1}-W_k^{2N+1} \qquad (-N\leq k\leq N), \qquad (5.3)$$

(5. 2b) can be rewritten as

$$W^N F_N(W)=\sum_{k=-N}^{N} f_k \frac{W^{2N+1}-W_k^{2N+1}}{(W-W_k)}\frac{W_k}{2N+1}. \qquad (5.4)$$

Next, writing

$$W^{2N+1}-W_k^{(2N+1)}=\exp(j(2N+1)x)-\exp(j(2N+1)k\frac{2\pi}{2N}$$

$$=\exp(j\frac{(2N+1)}{2}(x+k\frac{2\pi}{2N}))$$

$$\left(\exp(j\frac{(2N+1)}{2}(x-k\frac{2\pi}{2N}))-\exp(-j\frac{(2N+1)}{2}(x-k\frac{2\pi}{2N}))\right)$$

$$=2j\exp(j\frac{(2N+1)}{2}(x+k\frac{2\pi}{2N}))\sin\left(\frac{(2N+1)}{2}(x-k\frac{2\pi}{2N})\right), \qquad (5.5a)$$

$$W-W_k=2j\exp(j\frac{1}{2}(x+k\frac{2\pi}{2N}))\sin\left(\frac{1}{2}(x-k\frac{2\pi}{2N})\right), \qquad (5.5b)$$

and substituting these relations in (5. 4), we obtain, from (5. 2a) and (5.4),

$$F_N(W)=f_N(x)=\sum_{k=-N}^{N} f_k \frac{\sin\left(\frac{(2N+1)}{2}(x-k\frac{2\pi}{2N})\right)}{\sin\left(\frac{1}{2}(x-k\frac{2\pi}{2N})\right)}\frac{1}{2N+1}. \qquad (5.6)$$

We shall now prove

Theorem 4 : Let $f(x)$ be a periodic function with the period 2π of bounded variation, and let $f_N(x)$ be defined by (5. 6), provided that $f_k=f(x_k)$ at $x_k=k\frac{2\pi}{2N}$ $(-N\leq k\leq N)$. Then,

$$\lim_{N\to\infty} f_N(x)=f(x), \quad -\pi\leq k\leq\pi \qquad (5.7)$$

holds.

(Proof) We shall define a step function (10) by

$$\varphi_N(x)=f_k=f(k\frac{2\pi}{2N}),\ x\in I_k^{(N)}=[k\frac{2\pi}{2N},(k+1)\frac{2\pi}{2N}], \qquad (5.8a)$$

the discrete Dirichlet kernel by

$$D_N(x;k)=D_N(x-k\frac{2\pi}{2N})=\frac{\sin\left(\frac{(2N+1)}{2}(x-k\frac{2\pi}{2N})\right)}{\sin\left(\frac{1}{2}(x-k\frac{2\pi}{2N})\right)} \qquad (5.8b)$$

and a step function $\Delta_N(x;k)$ by

$$\frac{1}{2N+1}\Delta_N(x;k)=1 \text{ for } x\in I_k^{(N)}, \text{ and } =0 \text{ for } x\notin I_k^{(N)}. \qquad (5.8c)$$

Then, referring to Theorem 3, we see that $D_N(x-k\frac{2\pi}{2N})$ satisfies

$$\frac{1}{2N+1}D_N(x;k)=\delta_{ik},\ x=i\frac{2\pi}{2N}, \tag{5. 8d}$$

δ_{ik} being Kronecker 's symbol, and we may write

$$\varphi_N(x)=\sum_{k=-N}^{N}\varphi_N(x)\varDelta_N(x;k)\frac{1}{2N+1}$$
$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi_N(u)\varDelta_N(x;\left[\frac{2N}{2\pi}\right])du, \tag{5. 9}$$

in which the brackets means Gauss' symbol. Here it is noted that the right-hand side is Lebesgue integrable(10). Further, define an integral

$$\hat{f}_N(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(u)\frac{\sin\left(\frac{(2N+1)}{2}(x-u)\right)}{\sin\left(\frac{1}{2}(x-u)\right)}du, \tag{5. 10}$$

and write

$$f(x)-\hat{f}_N(x)=(f(x)-\varphi_N(x))-(\hat{f}_N(x)-\varphi_N(x)), \tag{5. 11a}$$
$$f_N(x)-\hat{f}_N(x)=(f_N(x)-\varphi_N(x))-(\hat{f}_N(x)-\varphi_N(x)). \tag{5. 11b}$$

Then, since $f(x)$ is of bounded variation by assumption, it can be shown that for arbitrary $\varepsilon>0$, there exists an integer N such that

$$|f(x)-\varphi_N(x)|<\varepsilon,\ -\pi\leq x\leq\pi \tag{5. 12}$$

holds. We may write, from (5. 9) and (5. 10), as

$$\hat{f}_N(x)-\varphi_N(x)=$$
$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(f(u)\frac{\sin\left(\frac{(2N+1)}{2}(u-x)\right)}{\sin\left(\frac{1}{2}(u-x)\right)}-\varphi_N(u)\varDelta_N(x;\left[\frac{2N}{2\pi}u\right])\right)du, \tag{5. 13}$$

and, from (5. 6),

$$f_N(x)-\varphi_N(x)=\sum_{k=-N}^{N}\varphi_N(x)(D_N(x;k)-\varDelta_N(x;k))\frac{1}{2N+1}. \tag{5. 14}$$

From (5. 13), we obtain

$$|\hat{f}_N(x)-\varphi_N(x)|\leq\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(u)-\varphi_N(u)|\left|\frac{\sin\left(\frac{(2N+1)}{2}(u-x)\right)}{\sin\left(\frac{1}{2}(u-x)\right)}\right|du$$
$$+\frac{1}{2\pi}\left|\int_{-\pi}^{\pi}\left(\frac{\sin\left(\frac{(2N+1)}{2}(u-x)\right)}{\sin\left(\frac{1}{2}(u-x)\right)}-\varDelta_N(x;\left[\frac{2N}{2\pi}u\right])\right)\varphi_N(u)\,du\right|. \tag{5. 15}$$

Applying Schwarz's inequality to the first term on the right-hand side of (5. 15), and considering (5. 12), we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(u)-\varphi_N(u)|\left|\frac{\sin\left(\frac{(2N+1)}{2}(u-x)\right)}{\sin\left(\frac{1}{2}(u-x)\right)}\right|du\leq$$
$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(u)-\varphi_N(u)|^2du\,\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{\sin\left(\frac{(2N+1)}{2}(u-x)\right)}{\sin\left(\frac{1}{2}(u-x)\right)}\right|^2du\right)^{\frac{1}{2}}$$
$$<\varepsilon. \tag{5. 16a}$$

Next, in the second term of the right-hand side of (5. 15), for arbitrarily chosen $\varepsilon'>0$, there exists a sufficiently large integer N such that

$$\frac{1}{2\pi}\left|\int_{-\pi}^{\pi}\left(\frac{\sin\left(\frac{(2N+1)}{2}(u-x)\right)}{\sin\left(\frac{1}{2}(u-x)\right)}-\Delta_N(x;\left[\frac{2N}{2\pi}u\right])\right)\varphi_N(u)\,du\right|<\varepsilon' \quad (5.\ 16b)$$

hold. From (5. 14), an inequality similar to (5. 16c) holds:

$$|f_N(x)-\varphi_N(x)|\leq\sum_{k=-N}^{N}|D_N(x;k)-\Delta_N(x;k)||\varphi_N(x)|\frac{1}{2N+1}. \quad (5.\ 16c)$$

Here, for an arbitrarily chosen $\varepsilon''>0$, if we choose a sufficiently large integer N, then

$$\frac{1}{2N+1}|D_N(x:k)-\Delta_N(x;k)|<\varepsilon''$$

holds, and, from (5. 16c), we have

$$|f_N(x)-\varphi_N(x)|\leq\varepsilon''\operatorname*{Max}_x|\varphi_N(x)|<K\varepsilon'', \quad (5.\ 16d)$$

K being a proper constant.

Next, since we have, from (5. 11a) and (5. 11b),

$$|f(x)-\hat{f}_N(x)|<|f(x)-\varphi_N(x)|+|\hat{f}_N(x)-\varphi_N(x)|, \quad (5.\ 17a)$$

and

$$|f_N(x)-\hat{f}_N(x)|<|f_N(x)-\varphi_N(x)|+|\hat{f}_N(x)-\varphi_N(x)|, \quad (5.\ 17b)$$

substituting (5. 16a), (5. 16b), and (5. 16d) in (5. 17a) and (5. 17b), we obtain, for sufficiently large integer N,

$$|f(x)-\hat{f}_N(x)|<\varepsilon+\varepsilon',\ |f_N(x)-\hat{f}_N(x)|<\varepsilon'+K\varepsilon'', \quad (5.\ 17c)$$

and hence,

$$\begin{aligned}&|f(x)-f_N(x)|\leq|f(x)-\hat{f}_N(x)|+|f_N(x)-\hat{f}_N(x)|\\&<\varepsilon'''=\varepsilon+2\varepsilon'+K\varepsilon''\end{aligned} \quad (5.\ 18)$$

where $\varepsilon'''>0$ can be chosen arbitrarily. We have thus proved Theorem 4. Q.E.D.

From the above-mentioned proof, we obtain the next corollary.

Corollary to Theorem 4 :

$$f(x)=\lim_{N\to\infty}\int_{-\pi}^{\pi}f(u)\frac{\sin\left(\frac{(2N+1)}{2}(x-u)\right)}{\sin\left(\frac{1}{2}(x-u)\right)}du. \quad (5.\ 19)$$

6. Concluding Remarks

Time series of physical signals can be classified as linear or nonlinear, stationary or nonstationary, according to characteristics of information sources from which the signals are delivered. No matter how their classification are, they can be represented by using the spectral distribution obtained by the Fourier transform as is mentioned in Section 1, and hence, how to model the information sources from the statistical data of signals, i.e., from their spectral distributions, is the fundamental problem of the signal processing. However, this problem is, in essence, still open and most desirarable to be solved.

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